

# BIMONADS AND HOPF MONADS ON CATEGORIES

BACHUKI MESABLISHVILI, TBILISI  
AND  
ROBERT WISBAUER, DÜSSELDORF

**ABSTRACT.** The purpose of this paper is to develop a theory of bimonads and Hopf monads on arbitrary categories thus providing the possibility to transfer the essentials of the theory of Hopf algebras in vector spaces to more general settings. There are several extensions of this theory to *monoidal* categories which in a certain sense follow the classical trace. Here we do not pose any conditions on our base category but we do refer to the monoidal structure of the category of endofunctors on any category  $\mathbb{A}$  and by this we retain some of the combinatorial complexity which makes the theory so interesting. As a basic tool we use distributive laws between monads and comonads (entwinings) on  $\mathbb{A}$ : we define a *bimonad* on  $\mathbb{A}$  as an endofunctor  $B$  which is a monad and a comonad with an entwining  $\lambda : BB \rightarrow BB$  satisfying certain conditions. This  $\lambda$  is also employed to define the category  $\mathbb{A}_B^B$  of (mixed)  $B$ -bimodules. In the classical situation, an entwining  $\lambda$  is derived from the twist map for vector spaces. Here this need not be the case but there may exist special distributive laws  $\tau : BB \rightarrow BB$  satisfying the Yang-Baxter equation (*local prebraidings*) which induce an entwining  $\lambda$  and lead to an extension of the theory of *braided Hopf algebras*.

An antipode is defined as a natural transformation  $S : B \rightarrow B$  with special properties and for categories  $\mathbb{A}$  with limits or colimits and bimonads  $B$  preserving them, the existence of an antipode is equivalent to  $B$  inducing an equivalence between  $\mathbb{A}$  and the category  $\mathbb{A}_B^B$  of  $B$ -bimodules. This is a general form of the *Fundamental Theorem* of Hopf algebras.

Finally we observe a nice symmetry: If  $B$  is an endofunctor with a right adjoint  $R$ , then  $B$  is a (Hopf) bimonad if and only if  $R$  is a (Hopf) bimonad. Thus a  $k$ -vector space  $H$  is a Hopf algebra if and only if  $\text{Hom}_k(H, -)$  is a Hopf bimonad. This provides a rich source for Hopf monads not defined by tensor products and generalises the well-known fact that a finite dimensional  $k$ -vector space  $H$  is a Hopf algebra if and only if its dual  $H^* = \text{Hom}_k(H, k)$  is a Hopf algebra. Moreover, we obtain that any set  $G$  is a group if and only if the functor  $\text{Map}(G, -)$  is a Hopf monad on the category of sets.

## CONTENTS

1. Introduction	1
2. Distributive laws	3
3. Actions on functors and Galois functors	6
4. Bimonads	12
5. Antipode	15
6. Local prebraidings for Hopf monads	18
7. Adjoints of bimonads	28
References	32

## 1. INTRODUCTION

The theory of algebras (monads) as well as of coalgebras (comonads) is well understood in various fields of mathematics as algebra (e.g. [8]), universal algebra (e.g. [13]), logic or operational semantics (e.g. [31]), theoretical computer science (e.g. [23]). The relationship between monads and comonads is controlled by *distributive laws* introduced in the seventies

by Beck (see [2]). In algebra one of the fundamental notions emerging in this context are the Hopf algebras. The definition is making heavy use of the tensor product and thus generalisations of this theory were mainly considered for *monoidal categories*. They allow readily to transfer formalisms from the category of vector spaces to the more general settings (e.g. Bespalov and Brabant [3] and [21]).

A Hopf algebra is an algebra as well as a coalgebra. Thus one way of generalisation is to consider distinct algebras and coalgebras and some relationship between them. This leads to the theory of *entwining structures* and *corings* over associative rings (e.g. [8]) and one may ask how to formulate this in more general categories. The definition of *bimonads* on a monoidal category as monads whose functor part is comonoidal by Bruguières and Virelizier in [7, 2.3] may be seen as going in this direction. Such functors are called *Hopf monads* in Moerdijk [22] and *opmonoidal monads* in McCrudden [18, Example 2.5]. In 2.2 we give more details of this notion.

Another extension of the theory of corings are the *generalised bialgebras* in Loday in [17]. These are Schur functors (on vector spaces) with a monad structure (operads) and a specified coalgebra structure satisfying certain compatibility conditions [17, 2.2.1]. While in [17] use is made of the canonical twist map, it is stressed in [7] that the theory is built up without reference to any braiding. More comments on these constructions are given in 2.3.

The purpose of the present paper is to formulate the essentials of the classical theory of Hopf algebras for any (not necessarily monoidal) category, thus making it accessible to a wide field of applications. We also employ the fact that the category of endofunctors (with the Godement product as composition) always has a tensor product given by composition of natural transformations but no tensor product is required for the base category.

Compatibility between monads and comonads are formulated as distributive laws whose properties are recalled in Section 2. In Section 3, general categorical notions are presented and *Galois functors* are defined and investigated, in particular equivalences induced for related categories (relative injectives).

As suggested in [33, 5.13], we define a *bimonad*  $\mathbf{H} = (H, m, e, \delta, \varepsilon)$  on any category  $\mathbb{A}$  as an endofunctor  $H$  with a monad and a comonad structure satisfying compatibility conditions (entwining) (see 4.1). The latter do not refer to any braiding but in special cases they can be derived from a *local prebraiding*  $\tau : HH \rightarrow HH$  (see 6.3). In this case the bimonad shows the characteristics of *braided bialgebras* (Section 6).

Related to a bimonad  $H$  there is the (Eilenberg-Moore) category  $\mathbb{A}_H^H$  of bimodules with a comparison functor  $K_H : \mathbb{A} \rightarrow \mathbb{A}_H^H$ . An *antipode* is defined as a natural transformation  $S : H \rightarrow H$  satisfying  $m \cdot SH \cdot \delta = e \cdot \varepsilon = m \cdot HS \cdot \delta$ . It exists if and only if the natural transformation  $\gamma := Hm \cdot \delta H : HH \rightarrow HH$  is an isomorphism. If the category  $\mathbb{A}$  is Cauchy complete and  $H$  preserves limits or colimits, the existence of an antipode is equivalent to the comparison functor being an equivalence (see 5.6). This is a general form of the Fundamental Theorem for Hopf algebras. Any generalisation of Hopf algebras should offer an extension of this important result.

Of course, bialgebras and Hopf algebras over commutative rings  $R$  provide the prototypes for this theory: on  $R\text{-Mod}$ , the category of  $R$ -modules, one considers the endofunctor  $B \otimes_R - : R\text{-Mod} \rightarrow R\text{-Mod}$  where  $B$  is an  $R$ -module with algebra and coalgebra structures, and an entwining derived from the twist map (braiding)  $M \otimes_R N \rightarrow N \otimes_R M$  (e.g. [5, Section 8]).

More generally, for a comonad  $H$ , the entwining  $\lambda : HH \rightarrow HH$  may be derived from a *local prebraiding*  $\tau : HH \rightarrow HH$  (see 6.7) and then results similar to those known for braided Hopf algebras are obtained. In particular, the composition  $HH$  is again a bimonad (see 6.8) and, if  $\tau^2 = 1$ , an *opposite bimonad* can be defined (see 6.10).

In case a bimonad  $H$  on  $\mathbb{A}$  has a right (or left) adjoint endofunctor  $R$ , then  $R$  is again a bimonad and has an antipode (or local prebraiding) if and only if so does  $H$  (see 7.5). In particular, for  $R$ -modules  $B$ , the functor  $\text{Hom}_R(B, -)$  is right adjoint to  $B \otimes_R -$  and hence  $B$  is a Hopf algebra if and only if  $\text{Hom}_R(B, -)$  is a Hopf monad. This provides a rich source for examples of Hopf monads not defined by a tensor product and extends a symmetry principle known for finite dimensional Hopf algebras (see 7.8). We close with the observation that a set  $G$  is a group if and only if the endofunctor  $\text{Map}(G, -)$  is a Hopf monad on the category of sets (7.9).

Note that the pattern of our definition of bimonads resembles the definition of *Frobenius monads* on any category by Street in [27]. Those are monads  $\mathbf{T} = (T, \mu, \eta)$  with natural transformations  $\varepsilon : T \rightarrow I$  and  $\rho : T \rightarrow TT$ , subject to suitable conditions, which induce a comonad structure  $\delta = T\mu \cdot \rho T : T \rightarrow TT$  and product and coproduct on  $T$  satisfy the compatibility condition  $T\mu \cdot \delta T = \delta \cdot \mu = \mu T \cdot T\delta$ .

## 2. DISTRIBUTIVE LAWS

Distributive laws between endofunctors were studied by Beck [2], Barr [1] and others in the seventies of the last century. They are a fundamental tool for us and we recall some facts needed in the sequel. For more details and references we refer to [33].

**2.1. Entwining from monad to comonad.** Let  $\mathbf{T} = (T, m, e)$  be a monad and  $\mathbf{G} = (G, \delta, \varepsilon)$  a comonad on a category  $\mathbb{A}$ . A natural transformation  $\lambda : TG \rightarrow GT$  is called a *mixed distributive law* or *entwining* from the monad  $\mathbf{T}$  to the comonad  $\mathbf{G}$  if the diagrams

$$\begin{array}{ccc}
 & G & \\
 eG \swarrow & & \searrow Ge \\
 TG & \xrightarrow{\lambda} & GT,
 \end{array}
 \quad
 \begin{array}{ccc}
 TG & \xrightarrow{\lambda} & GT \\
 T\varepsilon \searrow & & \swarrow \varepsilon T \\
 & T &
 \end{array}$$

$$\begin{array}{ccc}
 TG & \xrightarrow{T\delta} & TGG & \xrightarrow{\lambda G} & GTG \\
 \lambda \downarrow & & & & \downarrow G\lambda \\
 GT & \xrightarrow{\delta T} & GGT & & 
 \end{array}
 \quad \text{and} \quad
 \begin{array}{ccc}
 TTG & \xrightarrow{T\lambda} & TGT & \xrightarrow{\lambda T} & GTT \\
 mG \downarrow & & & & \downarrow Gm \\
 TG & \xrightarrow{\lambda} & GT & & 
 \end{array}$$

are commutative.

It is shown in [34] that for an arbitrary mixed distributive law  $\lambda : TG \rightarrow GT$  from a monad  $\mathbf{T}$  to a comonad  $\mathbf{G}$ , the triple  $\widehat{\mathbf{G}} = (\widehat{G}, \widehat{\delta}, \widehat{\varepsilon})$ , is a comonad on the category  $\mathbb{A}_{\mathbf{T}}$  of  $\mathbf{T}$ -modules (also called  $\mathbf{T}$ -algebras), where for any object  $(a, h_a)$  of  $\mathbb{A}_{\mathbf{T}}$ ,

- $\widehat{G}(a, h_a) = (G(a), G(h_a) \cdot \lambda_a)$ ;
- $(\widehat{\delta})_{(a, h_a)} = \delta_a$ , and
- $(\widehat{\varepsilon})_{(a, h_a)} = \varepsilon_a$ .

$\widehat{\mathbf{G}}$  is called the *lifting of  $\mathbf{G}$*  corresponding to the mixed distributive law  $\lambda$ .

Furthermore, the triple  $\widehat{\mathbf{T}} = (\widehat{T}, \widehat{m}, \widehat{e})$  is a monad on the category  $\mathbb{A}^{\mathbf{G}}$  of  $\mathbf{G}$ -comodules, where for any object  $(a, \theta_a)$  of the category  $\mathbb{A}^{\mathbf{G}}$ ,

- $\widehat{T}(a, \theta_a) = (T(a), \lambda_a \cdot T(\theta_a))$ ;
- $(\widehat{m})_{(a, \theta_a)} = m_a$ , and
- $(\widehat{e})_{(a, \theta_a)} = e_a$ .

This monad is called the *lifting of  $\mathbf{T}$*  corresponding to the mixed distributive law  $\lambda$ . One has an isomorphism of categories

$$(\mathbb{A}^G)_{\widehat{T}} \simeq (\mathbb{A}_T)^{\widehat{G}},$$

and we write  $\mathbb{A}_T^G(\lambda)$  for this category. An object of  $\mathbb{A}_T^G(\lambda)$  is a triple  $(a, h_a, \theta_a)$ , where  $(a, h_a) \in \mathbb{A}_T$  and  $(a, \theta_a) \in \mathbb{A}^G$  with commuting diagram

$$(2.1) \quad \begin{array}{ccccc} T(a) & \xrightarrow{h_a} & a & \xrightarrow{\theta_a} & G(a) \\ T(\theta_a) \downarrow & & & & \uparrow G(h_a) \\ TG(a) & \xrightarrow{\lambda_a} & GT(a). & & \end{array}$$

We consider two examples of entwining which may (also) be considered as generalisations of Hopf algebras. They are different from our approach and we will not refer to them later on.

**2.2. Opmonoidal functors.** Let  $(\mathbb{V}, \otimes, \mathbb{I})$  be a strict monoidal category. Following McCrudden [18, Example 2.5], one may call a monad  $(T, \mu, \eta)$  on  $\mathbb{V}$  *opmonoidal* if there exist morphisms

$$\theta : T(\mathbb{I}) \rightarrow \mathbb{I} \text{ and } \chi_{X,Y} : T(X \otimes Y) \rightarrow T(X) \otimes T(Y),$$

the latter natural in  $X, Y \in \mathbb{V}$ , which are compatible with the tensor structure of  $\mathbb{V}$  and the monad structure of  $T$ .

Such functors can also be characterised by the condition that the tensor product of  $\mathbb{V}$  can be lifted to the category of  $T$ -modules (e.g. [33, 3.4]). They were introduced and named *Hopf monads* by Moerdijk in [22, Definition 1.1] and called *bimonads* by Bruguières and Virelizier in [7, 2.3]. It is mentioned in [7, Example 2.8] that Szlachányi's bialgebroids in [29] may be interpreted in terms of such "bimonads". It is preferable to use the terminology from [18] since these functors are neither bimonads nor Hopf monads in a strict sense but rather an entwining (as in 2.1) between the monad  $T$  and the comonad  $T(\mathbb{I}) \otimes -$  on  $\mathbb{V}$ :

Indeed, the compatibility conditions required in the definitions induce a coproduct  $\chi_{\mathbb{I}, \mathbb{I}} : T(\mathbb{I}) \rightarrow T(\mathbb{I}) \otimes T(\mathbb{I})$  with counit  $\theta : T(\mathbb{I}) \rightarrow \mathbb{I}$ . Moreover, the relation between  $\chi$  and  $\mu$  (e.g. (15) in [7, 2.3]) lead to the commutative diagram (using  $X \otimes \mathbb{I} = X$ )

$$\begin{array}{ccccc} TT(X) & \xrightarrow{\mu} & T(X) & \xrightarrow{\chi_{\mathbb{I}, X}} & T(\mathbb{I}) \otimes T(X) \\ T(\chi_{\mathbb{I}, X}) \downarrow & & & & \uparrow T(\mathbb{I}) \otimes \mu_X \\ T(T(\mathbb{I}) \otimes T(X)) & \xrightarrow{\chi_{T(\mathbb{I}), T(X)}} & TT(\mathbb{I}) \otimes TT(X) & \xrightarrow{\mu_{\mathbb{I}} \otimes TT(X)} & T(\mathbb{I}) \otimes TT(X) \end{array}$$

This shows that  $T(X)$  is a mixed  $(T, T(\mathbb{I}) \otimes -)$ -bimodule for the entwining map

$$\lambda = (\mu_{\mathbb{I}} \otimes T(-)) \circ \chi_{T(\mathbb{I}), -} : T(T(\mathbb{I}) \otimes -) \rightarrow T(\mathbb{I}) \otimes T(-).$$

The *antipode* of a classical Hopf algebra  $H$  is defined as a special endomorphism of  $H$ . Since opmonoidal monads  $T$  relate two distinct functors it is not surprising that the notion of an antipode can not be transferred easily to this situation and the attempt to do so leads to an "apparently complicated definition" in [7, 3.3 and Remark 3.5]. Hereby the base category  $\mathcal{C}$  is required to be *autonomous*.

**2.3. Generalized bialgebras and Hopf operads.** The *generalised bialgebras* over fields as defined in Loday [17, Section 2.1] are similar to the mixed bimodules (see 2.1): they are vector spaces which are modules over some *operad*  $\mathcal{A}$  (Schur functors with multiplication and unit) and comodules over some coalgebras  $\mathcal{C}^c$ , which are linear duals of some operad  $\mathcal{C}$ .

Similar to the opmonoidal monads the coalgebraic structure is based on the tensor product (of vector spaces). The Hypothesis (H0) in [7] resembles the role of the entwining  $\lambda$  in 2.1. The Hypothesis (H1) requires that the free  $\mathcal{A}$ -algebra is a  $(\mathcal{C}^C, \mathcal{A})$ -bialgebra: this is similar to the condition on an  $A$ -coring  $C$ ,  $A$  an associative algebra, to have a  $C$ -comodule structure (equivalently the existence of a group-like element, e.g. [8, 28.2]). The condition (H2iso) plays the role of the canonical isomorphism defining *Galois corings* and the *Galois Coring Structure Theorem* [8, 28.19] may be compared with the *Rigidity Theorem* [17, 2.3.7]. The latter can be considered as a generalisation of the Hopf-Borel Theorem (see [17, 4.1.8]) and of the Cartier-Milnor-Moore Theorem (see [17, 4.1.3]). In [17, 3.2], *Hopf operads* are defined in the sense of Moerdijk [22] and thus the coalgebraic part is dependent on the tensor product. This is only a sketch of the similarities between Loday's setting and our approach here. It will be interesting to work out the relationship in more detail.

Similar to 2.1 we will also need the notion of mixed distributive laws from a comonad to a monad.

**2.4. Entwining from comonad to monad.** A natural transformation  $\lambda : GT \rightarrow TG$  is a *mixed distributive law* from a comonad  $\mathbf{G}$  to a monad  $\mathbf{T}$ , also called an *entwining* of  $\mathbf{G}$  and  $\mathbf{T}$ , if the diagrams

$$\begin{array}{ccc}
 & G & \\
 Ge \swarrow & & \searrow eG \\
 GT & \xrightarrow{\lambda} & TG, \\
 & & 
 \end{array}
 \quad
 \begin{array}{ccc}
 GT & \xrightarrow{\lambda} & TG \\
 \varepsilon T \searrow & & \swarrow T\varepsilon \\
 & T & 
 \end{array}$$

$$\begin{array}{ccccc}
 GTT & \xrightarrow{\lambda T} & TGT & \xrightarrow{T\lambda} & TTG \\
 Gm \downarrow & & & & \downarrow mG \\
 GT & \xrightarrow{\lambda} & TG, & & 
 \end{array}
 \quad
 \begin{array}{ccccc}
 GGT & \xrightarrow{G\lambda} & GTG & \xrightarrow{\lambda G} & TGG \\
 \delta T \uparrow & & & & \uparrow T\delta \\
 GT & \xrightarrow{\lambda} & TG & & 
 \end{array}$$

are commutative.

For convenience we recall the distributive laws between two monads and between two comonads (e.g. [2], [1], [33, 4.4 and 4.9]).

**2.5. Monad distributive.** Let  $\mathbf{F} = (F, m, e)$  and  $\mathbf{T} = (T, m', e')$  be monads on the category  $\mathbb{A}$ . A natural transformation  $\lambda : FT \rightarrow TF$  is said to be *monad distributive* if it induces the commutative diagrams

$$\begin{array}{ccc}
 & T & \\
 e_T \swarrow & & \searrow Te \\
 FT & \xrightarrow{\lambda} & TF, \\
 & & 
 \end{array}
 \quad
 \begin{array}{ccc}
 & F & \\
 Fe' \swarrow & & \searrow e'_F \\
 FT & \xrightarrow{\lambda} & TF. \\
 & & 
 \end{array}$$

$$\begin{array}{ccccc}
 FFT & \xrightarrow{m_T} & FT & & \\
 F\lambda \downarrow & & \downarrow \lambda & & \\
 FTF & \xrightarrow{\lambda_F} & TFF & \xrightarrow{Tm} & TF, \\
 & & & & 
 \end{array}
 \quad
 \begin{array}{ccccc}
 FTT & \xrightarrow{Fm'} & FT & & \\
 \lambda_T \downarrow & & \downarrow \lambda & & \\
 TFT & \xrightarrow{T\lambda} & TTF & \xrightarrow{m'_F} & TF. \\
 & & & & 
 \end{array}$$

In this case  $\lambda : FT \rightarrow TF$  induces a canonical monad structure on  $TF$ .

**2.6. Comonad distributive.** Let  $\mathbf{G} = (G, \delta, \varepsilon)$  and  $\mathbf{T} = (T, \delta', \varepsilon')$  be comonads on the category  $\mathbb{A}$ . A natural transformation  $\varphi : TG \rightarrow GT$  is said to be *comonad distributive* if it induces the commutative diagrams

$$\begin{array}{ccc}
TG & \xrightarrow{\varphi} & GT \\
& \searrow T\varepsilon & \swarrow \varepsilon_T \\
& T & 
\end{array}, \quad
\begin{array}{ccc}
TG & \xrightarrow{\varphi} & GT \\
& \searrow \varepsilon'_G & \swarrow G\varepsilon' \\
& G & 
\end{array},$$

$$\begin{array}{ccccc}
TG & \xrightarrow{T\delta} & TGG & \xrightarrow{\varphi_G} & GTG \\
\varphi \downarrow & & & & \downarrow G\varphi \\
GT & \xrightarrow{\delta_T} & GGT & & 
\end{array}, \quad
\begin{array}{ccccc}
TG & \xrightarrow{\delta'_G} & TTG & \xrightarrow{T\varphi} & TGT \\
\varphi \downarrow & & & & \downarrow \varphi_T \\
GT & \xrightarrow{G\delta'} & GTT & & 
\end{array}.$$

In this case  $\varphi: TG \rightarrow GT$  induces a canonical comonad structure on  $TG$ .

### 3. ACTIONS ON FUNCTORS AND GALOIS FUNCTORS

The language of modules over rings can also be used to describe actions of monads on functors. Doing this we define Galois functors and to characterise those we investigate the relationships between categories of relative injective objects.

**3.1.  $\mathbf{T}$ -actions on functors.** Let  $\mathbb{A}$  and  $\mathbb{B}$  be categories. Given a monad  $\mathbf{T} = (T, m, e)$  on  $\mathbb{A}$  and any functor  $L : \mathbb{A} \rightarrow \mathbb{B}$ , we say that  $L$  is a (*right*)  $\mathbf{T}$ -module if there exists a natural transformation  $\alpha_L : LT \rightarrow L$  such that the diagrams

$$(3.1) \quad
\begin{array}{ccc}
L & \xrightarrow{Le} & LT \\
& \searrow & \downarrow \alpha_L \\
& & L,
\end{array}
\quad
\begin{array}{ccc}
LTT & \xrightarrow{Lm} & LT \\
\alpha_L T \downarrow & & \downarrow \alpha_L \\
LT & \xrightarrow{\alpha_L} & L
\end{array}$$

commute. It is easy to see that  $(T, m)$  and  $(TT, Tm)$  both are  $\mathbf{T}$ -modules.

Similarly, given a comonad  $\mathbf{G} = (G, \delta, \varepsilon)$  on  $\mathbb{A}$ , a functor  $K : \mathbb{B} \rightarrow \mathbb{A}$  is a *left  $\mathbf{G}$ -comodule* if there exists a natural transformation  $\beta_K : K \rightarrow GK$  for which the diagrams

$$\begin{array}{ccc}
K & \xrightarrow{\beta_K} & GK \\
& \searrow & \downarrow \varepsilon K \\
& & K,
\end{array}
\quad
\begin{array}{ccc}
K & \xrightarrow{\beta_K} & GK \\
\beta_K \downarrow & & \downarrow \delta K \\
GK & \xrightarrow{G\beta_K} & GGK
\end{array}$$

commute.

Given two  $\mathbf{T}$ -modules  $(L, \alpha_L)$ ,  $(L', \alpha_{L'})$ , a natural transformation  $g : L \rightarrow L'$  is called  $\mathbf{T}$ -linear if the diagram

$$(3.2) \quad
\begin{array}{ccc}
LT & \xrightarrow{gT} & L'T \\
\alpha_L \downarrow & & \downarrow \alpha_{L'} \\
L & \xrightarrow{g} & L'
\end{array}$$

commutes.

**3.2. Lemma.** Let  $(L, \alpha_L)$  be a  $\mathbf{T}$ -module. If  $f, f' : TT \rightarrow L$  are  $\mathbf{T}$ -linear morphisms from the  $\mathbf{T}$ -module  $(TT, Tm)$  to the  $\mathbf{T}$ -module  $(L, \alpha_L)$  such that  $f \cdot Te = f' \cdot Te$ , then  $f = f'$ .

**Proof.** Since  $f \cdot Te = f' \cdot Te$ , we have  $\alpha_L \cdot fT \cdot TeT = \alpha_L \cdot f'T \cdot TeT$ . Moreover, since  $f$  and  $f'$  are both  $\mathbf{T}$ -linear, we have the commutative diagrams

$$\begin{array}{ccc} TTT & \xrightarrow{fT} & LT \\ Tm \downarrow & & \downarrow \alpha_L \\ TT & \xrightarrow{f} & L, \end{array} \quad \begin{array}{ccc} TTT & \xrightarrow{f'T} & LT \\ Tm \downarrow & & \downarrow \alpha_L \\ TT & \xrightarrow{f'} & L. \end{array}$$

Thus  $\alpha_L \cdot fT = f \cdot Tm$  and  $\alpha_L \cdot f'T = f' \cdot Tm$ , and we have  $f \cdot Tm \cdot TeT = f' \cdot Tm \cdot TeT$ . It follows - since  $Tm \cdot TeT = 1$  - that  $f = f'$ .  $\square$

**3.3. Left  $\mathbf{G}$ -comodule functors.** Let  $\mathbf{G}$  be a comonad on a category  $\mathbb{A}$ , let  $U^G : \mathbb{A}^G \rightarrow \mathbb{A}$  be the forgetful functor and write  $\phi^G : \mathbb{A} \rightarrow \mathbb{A}^G$  for the cofree  $\mathbf{G}$ -comodule functor. Fix a functor  $F : \mathbb{B} \rightarrow \mathbb{A}$ , and consider a functor  $\bar{F} : \mathbb{B} \rightarrow \mathbb{A}^G$  making the diagram

$$(3.3) \quad \begin{array}{ccc} \mathbb{B} & \xrightarrow{\bar{F}} & \mathbb{A}^G \\ & \searrow F & \swarrow U^G \\ & \mathbb{A} & \end{array}$$

commutative. Then  $\bar{F}(b) = (F(b), \alpha_{F(b)})$  for some  $\alpha_{F(b)} : F(b) \rightarrow GF(b)$ . Consider the natural transformation

$$(3.4) \quad \bar{\alpha}_F : F \rightarrow GF,$$

whose  $b$ -component is  $\alpha_{F(b)}$ . It should be pointed out that  $\bar{\alpha}_F$  makes  $F$  a left  $\mathbf{G}$ -comodule, and it is easy to see that there is a one to one correspondence between functors  $\bar{F} : \mathbb{B} \rightarrow \mathbb{A}^G$  making the diagram (3.3) commute and natural transformations  $\bar{\alpha}_F : F \rightarrow GF$  making  $F$  a left  $\mathbf{G}$ -comodule.

The following is an immediate consequence of (the dual of) [10, Propositions II,1.1 and II,1.4]:

**3.4. Theorem.** Suppose that  $F$  has a right adjoint  $R : \mathbb{A} \rightarrow \mathbb{B}$  with unit  $\eta : 1 \rightarrow RF$  and counit  $\varepsilon : FR \rightarrow 1$ . Then the composite

$$t_{\bar{F}} : FR \xrightarrow{\bar{\alpha}_F R} GFR \xrightarrow{G\varepsilon} G.$$

is a morphism from the comonad  $\mathbf{G}' = (FR, F\eta R, \varepsilon)$  generated by the adjunction  $\eta, \varepsilon : F \dashv R : \mathbb{A} \rightarrow \mathbb{B}$  to the comonad  $\mathbf{G}$ . Moreover, the assignment

$$\bar{F} \longrightarrow t_{\bar{F}}$$

yields a one to one correspondence between functors  $\bar{F} : \mathbb{B} \rightarrow \mathbb{A}^G$  making the diagram (3.3) commutative and morphisms of comonads  $t_{\bar{F}} : \mathbf{G}' \rightarrow \mathbf{G}$ .

**3.5. Definition.** We say that a left  $\mathbf{G}$ -comodule  $F : \mathbb{B} \rightarrow \mathbb{A}$  with a right adjoint  $R : \mathbb{B} \rightarrow \mathbb{A}$  is  $\mathbf{G}$ -Galois if the corresponding morphism  $t_{\bar{F}} : FR \rightarrow \mathbf{G}$  of comonads on  $\mathbb{A}$  is an isomorphism.

As an example, consider an  $A$ -coring  $\mathcal{C}$ ,  $A$  an associative ring, and any right  $\mathcal{C}$ -comodule  $P$  with  $S = \text{End}^{\mathcal{C}}(P)$ . Then there is a natural transformation

$$\tilde{\mu} : \text{Hom}_A(P, -) \otimes_S P \rightarrow - \otimes_A \mathcal{C}$$

and  $P$  is called a *Galois comodule* provided  $\tilde{\mu}_X$  is an isomorphism for any right  $A$ -module  $X$ , that is, the functor  $- \otimes_S P : \mathbb{M}_S \rightarrow \mathbb{M}^{\mathcal{C}}$  is a  $- \otimes_A \mathcal{C}$ -Galois comodule (see [32, Definiton 4.1]).

**3.6. Right adjoint functor of  $\overline{F}$ .** When the category  $\mathbb{B}$  has equalisers, the functor  $\overline{F}$  has a right adjoint, which can be described as follows: Writing  $\beta_R$  for the composite

$$R \xrightarrow{\eta_R} RFR \xrightarrow{Rt_{\overline{F}}} RG,$$

it is not hard to see that the equaliser  $(\overline{R}, \overline{e})$  of the following diagram

$$RU^G \xrightleftharpoons[\beta_R U^G]{RU^G \eta_G} RG U^G = RU^G \phi^G U^G,$$

where  $\eta_G : 1 \rightarrow \phi^G U^G$  is the unit of the adjunction  $U^G \dashv \phi^G$ , is right adjoint to  $\overline{F}$ .

**3.7. Adjoints and monads.** For categories  $\mathbb{A}, \mathbb{B}$ , let  $L : \mathbb{A} \rightarrow \mathbb{B}$  be a functor with right adjoint  $R : \mathbb{B} \rightarrow \mathbb{A}$ . Let  $\mathbf{T} = (T, m, e)$  be a monad on  $\mathbb{A}$  and suppose there exists a functor  $\overline{R} : \mathbb{B} \rightarrow \mathbb{A}_T$  yielding the commutative diagram

$$\begin{array}{ccc} \mathbb{B} & \xrightarrow{\overline{R}} & \mathbb{A}_T \\ & \searrow R & \swarrow U_T \\ & \mathbb{A} & \end{array}$$

Then  $\overline{R}(b) = (R(b), \beta_b)$  for some  $\beta_b : TR(b) \rightarrow R(b)$  and the collection  $\{\beta_b, b \in \mathbb{B}\}$  constitutes a natural transformation  $\beta_{\overline{R}} : TR \rightarrow R$ . It is proved in [10] that the natural transformation

$$t_{\overline{R}} : T \xrightarrow{T\eta} TRL \xrightarrow{\beta L} RL$$

is a morphism of monads. By the dual of [21, Theorem 4.4], we obtain:

*The functor  $\overline{R}$  is an equivalence of categories iff the functor  $R$  is monadic and  $t_{\overline{R}}$  is an isomorphism of monads.*

In view of the characterisation of Galois functors we have a closer look at some related classes of relative injective objects.

Let  $F : \mathbb{B} \rightarrow \mathbb{A}$  be any functor. Recall (from [14]) that an object  $b \in \mathbb{B}$  is said to be *F-injective* if for any diagram in  $\mathbb{B}$ ,

$$\begin{array}{ccc} b_1 & \xrightarrow{f} & b_2 \\ g \downarrow & \swarrow h & \\ & b & \end{array}$$

with  $F(f)$  a split monomorphism in  $\mathbb{A}$ , there exists a morphism  $h : b_2 \rightarrow b$  such that  $hf = g$ . We write  $\mathbf{Inj}(F, \mathbb{B})$  for the full subcategory of  $\mathbb{B}$  with objects all *F-injectives*.

The following result from [26] will be needed.

**3.8. Proposition.** *Let  $\eta, \varepsilon : F \dashv R : \mathbb{A} \rightarrow \mathbb{B}$  be an adjunction. For any object  $b \in \mathbb{B}$ , the following assertions are equivalent:*

- (a) *b is F-injective;*
- (b) *b is a coretract for some  $R(a)$ , with  $a \in \mathbb{A}$ ;*
- (c) *the b-component  $\eta_b : b \rightarrow RF(b)$  of  $\eta$  is a split monomorphism.*

**3.9. Remark.** For any  $a \in \mathbb{A}$ ,  $R(\varepsilon_a) \cdot \eta_{R(a)} = 1$  by one of the triangular identities for the adjunction  $F \dashv R$ . Thus,  $R(a) \in \mathbf{Inj}(F, \mathbb{B})$  for all  $a \in \mathbb{A}$ . Moreover, since the composite of coretracts is again a coretract, it follows from (b) that  $\mathbf{Inj}(F, \mathbb{B})$  is closed under coretracts.



**3.10. Functor between injectives.** Let  $K_{G'} : \mathbb{B} \rightarrow \mathbb{A}^{G'}$  be the comparison functor (notation as in 3.4). If  $b \in \mathbb{B}$  is  $F$ -injective, then  $K_{G'}(b) = (F(b), F(\eta_b))$  is  $U_{G'}$ -injective, since by the fact that  $\eta_b$  is a split monomorphism in  $\mathbb{B}$ ,  $(\eta_{G'})_{\phi_{G'}(b)} = F(\eta_b)$  is a split monomorphism in  $\mathbb{A}^{G'}$  ( $G'$  as in 3.4). Thus the functor  $K_{G'} : \mathbb{B} \rightarrow \mathbb{A}^{G'}$  yields a functor

$$\mathbf{Inj}(K_{G'}) : \mathbf{Inj}(\mathbf{F}, \mathbb{B}) \rightarrow \mathbf{Inj}(\phi^{G'}, \mathbb{A}^{G'}).$$

When  $\mathbb{B}$  has equalisers, this functor is an equivalence of categories (see [26]).

We shall henceforth assume that  $\mathbb{B}$  has equalisers.

**3.11. Proposition.** *The functor  $\overline{R} : \mathbb{A}^G \rightarrow \mathbb{B}$  restricts to a functor*

$$\overline{R}' : \mathbf{Inj}(U^G, \mathbb{A}^G) \rightarrow \mathbf{Inj}(F, \mathbb{B}).$$

**Proof.** Let  $(a, \theta_a)$  be an arbitrary object of  $\mathbf{Inj}(U^G, \mathbb{A}^G)$ . Then, by Proposition 3.8, there exists an object  $a_0 \in \mathbb{A}$  such that  $(a, \theta_a)$  is a coretraction of  $\phi^G(a_0) = (G(a_0), \delta_{a_0})$  in  $\mathbb{A}^G$ , i.e., there exist morphisms

$$f : (a, \theta_a) \rightarrow (G(a_0), \delta_{a_0}) \text{ and } g : (G(a_0), \delta_{a_0}) \rightarrow (a, \theta_a)$$

in  $\mathbb{A}^G$  with  $gf = 1$ . Since  $f$  and  $g$  are morphisms in  $\mathbb{A}^G$ , the diagram

$$\begin{array}{ccc} G(a_0) & \xrightarrow{(\delta_G)_{a_0}} & GG(a_0) \\ f \updownarrow g & & G(f) \updownarrow G(g) \\ a & \xrightarrow{\theta_a} & G(a) \end{array}$$

commutes. By naturality of  $\beta_R$ , the diagram

$$\begin{array}{ccc} RG(a_0) & \xrightarrow{(\beta_R)_{G(a_0)}} & RGG(a_0) \\ R(f) \updownarrow R(g) & & RG(f) \updownarrow RG(g) \\ R(a) & \xrightarrow{(\beta_R)_a} & RG(a) \end{array}$$

also commutes. Consider now the following commutative diagram

$$(3.5) \quad \begin{array}{ccccc} R(a_0) & \xrightarrow{\beta_{a_0}} & RG(a_0) & \xrightleftharpoons[R((\delta_G)_{a_0})]{(\beta_R)_{G(a_0)}} & RGG(a_0) \\ \vdots \downarrow & & \updownarrow R(f) \quad \updownarrow R(g) & & \updownarrow RG(f) \quad \updownarrow RG(g) \\ \overline{R}(a, \theta_a) & \xrightarrow{\overline{e}_{(a, \theta_a)}} & R(a) & \xrightleftharpoons[R(\theta_a)]{(\beta_R)_a} & RG(a). \end{array}$$

It is not hard to see that the top row of this diagram is a (split) equaliser (see, [12]), and since the bottom row is an equaliser by the very definition of  $\overline{e}$ , it follows from the commutativity of the diagram that  $\overline{R}(a, \theta_a)$  is a coretract of  $R(a_0)$ , and thus is an object of  $\mathbf{Inj}(F, \mathbb{B})$  (see Remark 3.9). It means that the functor  $\overline{R} : \mathbb{A}^G \rightarrow \mathbb{B}$  can be restricted to a functor  $\overline{R}' : \mathbf{Inj}(U^G, \mathbb{A}^G) \rightarrow \mathbf{Inj}(F, \mathbb{B})$ .  $\square$

**3.12. Proposition.** *Suppose that for any  $b \in \mathbb{B}$ ,  $(t_{\overline{F}})_{F(b)}$  is an isomorphism. Then the functor  $\overline{F} : \mathbb{B} \rightarrow \mathbb{A}^G$  can be restricted to a functor*

$$\overline{F}' : \mathbf{Inj}(F, \mathbb{B}) \rightarrow \mathbf{Inj}(U^G, \mathbb{A}^G).$$

**Proof.** Let  $\delta'$  denote the comultiplication in the comonad  $\mathbf{G}'$  (see 3.4), i.e.,  $\delta' = F\eta R$ . Then for any  $b \in \mathbb{B}$ ,

$$\begin{aligned}\overline{F}(RF(b)) &= \mathbb{A}_{t_{\overline{F}}}(\phi^{G'}(UF(b))) = \mathbb{A}_{t_{\overline{F}}}(FRF(b), F\eta_{RF(b)}) \\ &= \mathbb{A}_{t_{\overline{F}}}(G'F(b), \delta'_{F(b)}) = (G'F(b), (t_{\overline{F}})_{G'F(b)} \cdot \delta'_{F(b)}).\end{aligned}$$

Consider now the diagram

$$\begin{array}{ccc} G'F(b) & \xrightarrow{(t_{\overline{F}})_{F(b)}} & GF(b) \\ \delta'_{F(b)} \downarrow & (1) & \downarrow \delta_{F(b)} \\ G'G'F(b) & \searrow (t_{\overline{F}})_{F(b)} \cdot (t_{\overline{F}})_{F(b)} & \\ (t_{\overline{F}})_{G'F(b)} \downarrow & & \downarrow \\ GG'F(b) & \xrightarrow{G((t_{\overline{F}})_{F(b)})} & GGF(b), \end{array}$$

in which the triangle commutes by the definition of the composite  $(t_{\overline{F}})_{F(b)} \cdot (t_{\overline{F}})_{F(b)}$ , while the diagram (1) commutes since  $t_{\overline{F}}$  is a morphism of comonads. The commutativity of the outer diagram shows that  $(t_{\overline{F}})_{F(b)}$  is a morphism from the  $G$ -coalgebra  $\overline{F}(RF(b)) = (G'F(b), (t_{\overline{F}})_{G'F(b)} \cdot \delta'_{F(b)})$  to the  $G$ -coalgebra  $(GF(b), \delta_{F(b)})$ . Moreover,  $(t_{\overline{F}})_{F(b)}$  is an isomorphism by our assumption. Thus, for any  $b \in \mathbb{B}$ ,  $\overline{F}(RF(b))$  is isomorphic to the  $G$ -coalgebra  $(GF(b), \delta_{F(b)})$ , which is of course an object of the category  $\mathbf{Inj}(U^G, \mathbb{A}^G)$ . Now, since any  $b \in \mathbf{Inj}(F, \mathbb{B})$  is a coretract of  $RF(b)$  (see Remark 3.9), and since any functor takes coretracts to coretracts, it follows that, for any  $b \in \mathbf{Inj}(F, \mathbb{B})$ ,  $\overline{F}(b)$  is a coretract of the  $G$ -coalgebra  $(GF(b), \delta_{F(b)}) \in \mathbf{Inj}(U^G, \mathbb{A}^G)$ , and thus is an object of the category  $\mathbf{Inj}(U^G, \mathbb{A}^G)$  again by Remark 3.9. This completes the proof.  $\square$

The following technical observation is needed for the next proposition.

**3.13. Lemma.** *Let  $\iota, \kappa : W \dashv W' : \mathbb{Y} \rightarrow \mathbb{X}$  be an adjunction of any categories. If  $i : x' \rightarrow x$  and  $j : x \rightarrow x'$  are morphisms in  $\mathbb{X}$  such that  $ji = 1$  and if  $\iota_x$  is an isomorphism, then  $\iota_{x'}$  is also an isomorphism.*

**Proof.** Since  $ji = 1$ , the diagram

$$x' \xrightarrow{i} x \xrightleftharpoons[ij]{1} x$$

is a split equaliser. Then the diagram

$$W'W(x') \xrightarrow{W'W(i)} W'W(x) \xrightleftharpoons[W'W(ij)]{1} W'W(x)$$

is also a split equaliser. Now considering the following commutative diagram

$$\begin{array}{ccccc} x' & \xrightarrow{i} & x & \xrightleftharpoons[ij]{1} & x \\ \downarrow \iota_{x'} & & \downarrow \kappa_x & & \downarrow \kappa_x \\ W'W(x') & \xrightarrow{W'W(i)} & W'W(x) & \xrightleftharpoons[W'W(ij)]{1} & W'W(x) \end{array}$$

and recalling that the vertical two morphisms are both isomorphisms by assumption, we get that the morphism  $\iota_{x'}$  is also an isomorphism.  $\square$

**3.14. Proposition.** *In the situation of Proposition 3.12,  $\mathbf{Inj}(F, \mathbb{B})$  is (isomorphic to) a coreflective subcategory of the category  $\mathbf{Inj}(U^G, \mathbb{A}^G)$ .*

**Proof.** By Proposition 3.11, the functor  $\overline{R}$  restricts to a functor

$$\overline{R}' : \mathbf{Inj}(U^G, \mathbb{A}^G) \rightarrow \mathbf{Inj}(F, \mathbb{B}),$$

while according to Proposition 3.12, the functor  $\overline{F}$  restricts to a functor

$$\overline{F}' : \mathbf{Inj}(F, \mathbb{B}) \rightarrow \mathbf{Inj}(U^G, \mathbb{A}^G).$$

Since

- $\overline{F}$  is a left adjoint to  $\overline{R}$ ,
- $\mathbf{Inj}(F, \mathbb{B})$  is a full subcategory of  $\mathbb{B}$ , and
- $\mathbf{Inj}(U^G, \mathbb{A}^G)$  is a full subcategory of  $\mathbb{A}^G$ ,

the functor  $\overline{F}'$  is left adjoint to the functor  $\overline{R}'$ , and the unit  $\overline{\eta}' : 1 \rightarrow \overline{R}'\overline{F}'$  of the adjunction  $\overline{F}' \dashv \overline{R}'$  is the restriction of  $\overline{\eta} : \overline{F} \dashv \overline{R}$  to the subcategory  $\mathbf{Inj}(F, \mathbb{B})$ , while the counit  $\overline{\varepsilon}' : \overline{F}'\overline{R}' \rightarrow 1$  of this adjunction is the restriction of  $\overline{\varepsilon} : \overline{F}\overline{R} \rightarrow 1$  to the subcategory  $\mathbf{Inj}(U^G, \mathbb{A}^G)$ .

Next, since the top of the diagram 3.5 is a (split) equaliser,  $\overline{R}(G(a_0), \delta_{a_0}) \simeq R(a_0)$ . In particular, taking  $(GF(b), \delta_{F(b)})$ , we see that

$$RF(b) \simeq \overline{R}(GF(b), \delta_{F(b)}) = \overline{R}\overline{F}(UF(b)).$$

Thus, the  $RF(b)$ -component  $\overline{\eta}'_{RF(b)}$  of the unit  $\overline{\eta}' : 1 \rightarrow \overline{R}'\overline{F}'$  of the adjunction  $\overline{F}' \dashv \overline{R}'$  is an isomorphism. It now follows from Lemma 3.13 - since any  $b \in \mathbf{Inj}(F, \mathbb{B})$  is a coretraction of  $RF(b)$  - that  $\overline{\eta}'_b$  is an isomorphism for all  $b \in \mathbf{Inj}(F, \mathbb{B})$  proving that the unit  $\overline{\eta}'$  of the adjunction  $\overline{F}' \dashv \overline{R}'$  is an isomorphism. Thus  $\mathbf{Inj}(F, \mathbb{B})$  is (isomorphic to) a coreflective subcategory of the category  $\mathbf{Inj}(U^G, \mathbb{A}^G)$ .  $\square$

**3.15. Corollary.** *In the situation of Proposition 3.12, suppose that each component of the unit  $\eta : 1 \rightarrow RF$  is a split monomorphism. Then the category  $\mathbb{B}$  is (isomorphic to) a coreflective subcategory of  $\mathbf{Inj}(U^G, \mathbb{A}^G)$ .*

**Proof.** When each component of the unit  $\eta : 1 \rightarrow RF$  is a split monomorphism, it follows from Proposition 3.8 that every  $b \in \mathbb{B}$  is  $F$ -injective; i.e.  $\mathbb{B} = \mathbf{Inj}(F, \mathbb{B})$ . The assertion now follows from Proposition 3.14.  $\square$

**3.16. Characterisation of  $\mathbf{G}$ -Galois comodules.** *Assume  $\mathbb{B}$  to admit equalisers, let  $\mathbf{G}$  be a comonad on  $\mathbb{A}$ , and  $F : \mathbb{B} \rightarrow \mathbb{A}$  a functor with right adjoint  $R : \mathbb{A} \rightarrow \mathbb{B}$ . If there exists a functor  $\overline{F} : \mathbb{A} \rightarrow \mathbb{A}^G$  with  $U^G\overline{F} = F$ , then the following are equivalent:*

- (a)  $F$  is  $\mathbf{G}$ -Galois, i.e.  $t_{\overline{F}} : \mathbf{G}' \rightarrow \mathbf{G}$  is an isomorphism;
- (b) the following composite is an isomorphism:

$$\overline{F}R \xrightarrow{\eta_G\overline{F}R} \phi^G U^G \overline{F}R = \phi^G FR \xrightarrow{\phi^G \varepsilon} \phi^G ;$$

- (c) the functor  $\overline{F} : \mathbb{B} \rightarrow \mathbb{A}^G$  restricts to an equivalence of categories

$$\mathbf{Inj}(F, \mathbb{B}) \rightarrow \mathbf{Inj}(U^G, \mathbb{A}^G);$$

- (d) for any  $(a, \theta_a) \in \mathbf{Inj}(U^G, \mathbb{A}^G)$ , the  $(a, \theta_a)$ -component  $\overline{\varepsilon}_{(a, \theta_a)}$  of the counit  $\overline{\varepsilon}$  of the adjunction  $\overline{F} \dashv \overline{R}$ , is an isomorphism;

(e) for any  $a \in \mathbb{A}$ ,  $\bar{\varepsilon}_{\phi_G(a)} = \bar{\varepsilon}_{(G(a), \delta_a)}$  is an isomorphism.

**Proof.** That (a) and (b) are equivalent is proved in [9]. By the proof of [12, Theorem of 2.6], for any  $a \in \mathbb{A}$ ,  $\bar{\varepsilon}_{\phi^G(a)} = \bar{\varepsilon}_{(G(a), \delta_a)} = (t_{\bar{F}})_a$ , thus (a) and (e) are equivalent.

By Remark 3.9, (d) implies (e).

Since  $\mathbb{B}$  admits equalisers by our assumption on  $\mathbb{B}$ , it follows from Proposition 3.10 that the functor  $\mathbf{Inj}(K_{G'})$  is an equivalence of categories. Now, if  $t_{\bar{F}} : \mathbf{G}' \rightarrow \mathbf{G}$  is an isomorphism of comonads, then the functor  $\mathbb{A}_{t_{\bar{F}}}$  is an isomorphism of categories, and thus  $\bar{F}$  is isomorphic to the comparison functor  $K_{G'}$ . It now follows from Proposition 3.10 that  $\bar{F}$  restricts to the functor  $\mathbf{Inj}(F, \mathbb{B}) \rightarrow \mathbf{Inj}(U^G, \mathbb{A}^G)$  which is an equivalence of categories. Thus (a)  $\Rightarrow$  (c).

If the functor  $\bar{F} : \mathbb{B} \rightarrow \mathbb{A}^G$  restricts to a functor

$$\bar{F}' : \mathbf{Inj}(F, \mathbb{B}) \rightarrow \mathbf{Inj}(U^G, \mathbb{A}^G),$$

then one can prove as in the proof of Proposition 3.9 that  $\bar{F}'$  is left adjoint to  $\bar{R}'$  and that the counit  $\bar{\varepsilon}' : \bar{F}' \bar{R}' \rightarrow 1$  of this adjunction is the restriction of the counit  $\bar{\varepsilon} : \bar{F} \bar{R} \rightarrow 1$  of the adjunction  $\bar{F} \dashv \bar{R}$  to the subcategory  $\mathbf{Inj}(U^G, \mathbb{A}^G)$ . Now, if  $\bar{F}'$  is an equivalence of categories, then  $\bar{\varepsilon}'$  is an isomorphism. Thus, for any  $(a, \theta_a) \in \mathbf{Inj}(U^G, \mathbb{A}^G)$ ,  $\bar{\varepsilon}'_{(a, \theta_a)}$  is an isomorphism proving that (c)  $\Rightarrow$  (d).  $\square$

#### 4. BIMONADS

The following definition was suggested in [33, 5.13]. For monoidal categories similar conditions were considered by Takeuchi [30, Definition 5.1] and in [21]. Notice that the term *bimonad* is used with a different meaning in by Bruguières and Virelizier (see 2.2).

**4.1. Definition.** A *bimonad*  $\mathbf{H}$  on a category  $\mathbb{A}$  is an endofunctor  $H : \mathbb{A} \rightarrow \mathbb{A}$  which has a monad structure  $\underline{H} = (H, m, e)$  and a comonad structure  $\overline{H} = (H, \delta, \varepsilon)$  such that

- (i)  $\varepsilon : H \rightarrow 1$  is a morphism from the monad  $\underline{H}$  to the identity monad;
- (ii)  $e : 1 \rightarrow H$  is a morphism from the identity comonad to the comonad  $\overline{H}$ ;
- (iii) there is a mixed distributive law  $\lambda : HH \rightarrow HH$  from the monad  $\underline{H}$  to the comonad  $\overline{H}$  yielding the commutative diagram

$$(4.1) \quad \begin{array}{ccccc} HH & \xrightarrow{m} & H & \xrightarrow{\delta} & HH \\ H\delta \downarrow & & & & \uparrow Hm \\ HHH & \xrightarrow{\lambda H} & HHH & & \end{array}$$

Note that the conditions (i), (ii) just mean commutativity of the diagrams

$$(4.2) \quad \begin{array}{ccc} HH & \xrightarrow{H\varepsilon} & H \\ m \downarrow & & \downarrow \varepsilon \\ H & \xrightarrow{\varepsilon} & 1, \end{array} \quad \begin{array}{ccc} 1 & \xrightarrow{e} & H \\ e \downarrow & & \downarrow \delta \\ H & \xrightarrow{eH} & HH \end{array}, \quad \begin{array}{ccc} 1 & \xrightarrow{e} & H \\ & \searrow = & \downarrow \varepsilon \\ & & 1. \end{array}$$

**4.2. Hopf modules.** Given a bimonad  $\mathbf{H} = (\underline{H}, \overline{H}, \lambda)$  on  $\mathbb{A}$ , the objects of  $\mathbb{A}_{\underline{H}}^{\overline{H}}(\lambda)$  are called *mixed  $H$ -bimodules* or  *$H$ -Hopf modules*. By 2.1, they are triples  $(a, h_a, \theta_a)$ , where  $(a, h_a) \in \mathbb{A}_{\underline{H}}$  and  $(a, \theta_a) \in \mathbb{A}^{\overline{H}}$  with commuting diagram

$$(4.3) \quad \begin{array}{ccccc} \underline{H}(a) & \xrightarrow{h_a} & a & \xrightarrow{\theta_a} & \overline{H}(a) \\ \underline{H}(\theta_a) \downarrow & & & & \uparrow \overline{H}(h_a) \\ \underline{H}\overline{H}(a) & \xrightarrow{\lambda_a} & \overline{H}\underline{H}(a). & & \end{array}$$

The morphisms in  $\mathbb{A}_{\underline{H}}^{\overline{H}}(\lambda)$  are morphisms in  $\mathbb{A}$  which are  $\underline{H}$ -monad as well as  $\overline{H}$ -comonad morphisms,

Recall that a morphism  $q : a \rightarrow a$  in a category  $\mathbb{A}$  is an *idempotent* when  $qq = q$ , and an idempotent  $q$  is said to *split* if  $q$  has a factorization  $q = i \cdot \bar{q}$  with  $\bar{q} \cdot i = 1$ . This happens if and only if the equaliser  $i = \text{Eq}(1_a, q)$  exists or - equivalently - the coequaliser  $\bar{q} = \text{Coeq}(1_a, q)$  exists (e.g. [6, Proposition 1]). The category  $\mathbb{A}$  is called *Cauchy complete* provided every idempotent in  $\mathbb{A}$  splits.

**4.3. Comparison functors.** Given a bimonad  $\mathbf{H} = (\underline{H} = (H, m, e), \overline{H} = (H, \delta, \varepsilon), \lambda)$  on a category  $\mathbb{A}$ , the mixed distributive law  $\lambda$  induces functors

$$\begin{aligned} K_{\underline{H}} : \mathbb{A} &\rightarrow (\mathbb{A}_{\underline{H}})^{\widehat{\overline{H}}}, & a &\mapsto ((H(a), m_a), \delta_a), \\ K_{\overline{H}} : \mathbb{A} &\rightarrow (\mathbb{A}^{\overline{H}})_{\widehat{\underline{H}}}, & a &\mapsto ((H(a), \delta_a), m_a), \end{aligned}$$

where  $\widehat{\overline{H}}$  is the lifting of the comonad  $\overline{H}$  and  $\widehat{\underline{H}}$  is the lifting of the monad  $\underline{H}$  by the mixed distributive law  $\lambda$ . We know that  $(\mathbb{A}_{\underline{H}})^{\widehat{\overline{H}}} \simeq (\mathbb{A}^{\overline{H}})_{\widehat{\underline{H}}}$  and denote this category by  $\mathbb{A}_{\underline{H}}^{\overline{H}}(\lambda)$  (see 2.1). There are commutative diagrams

$$(4.4) \quad \begin{array}{ccc} \mathbb{A} & \xrightarrow{K_{\underline{H}}} & (\mathbb{A}_{\underline{H}})^{\widehat{\overline{H}}} \\ & \searrow \phi_{\underline{H}} & \downarrow U_{\widehat{\overline{H}}} \\ & & \mathbb{A}_{\underline{H}}, \end{array} \quad \begin{array}{ccc} \mathbb{A} & \xrightarrow{K_{\overline{H}}} & (\mathbb{A}^{\overline{H}})_{\widehat{\underline{H}}} \\ & \searrow \phi_{\overline{H}} & \downarrow U_{\widehat{\underline{H}}} \\ & & \mathbb{A}^{\overline{H}}. \end{array}$$

(i) **The functor  $\phi_{\underline{H}}$ .** The forgetful functor  $U_{\underline{H}} : \mathbb{A}_{\underline{H}} \rightarrow \mathbb{A}$  is right adjoint to the free functor  $\phi_{\underline{H}}$  and the unit  $\eta_{\underline{H}} : 1 \rightarrow U_{\underline{H}}\phi_{\underline{H}}$  of this adjunction is the natural transformation  $e : 1 \rightarrow \underline{H}$ . Since  $\varepsilon : H \rightarrow 1$  is a morphism from the monad  $\underline{H}$  to the identity monad,  $\varepsilon \cdot e = 1$ , thus  $e$  is a split monomorphism.

The adjunction  $\phi_{\underline{H}} \dashv U_{\underline{H}}$  generates the comonad  $\phi_{\underline{H}}U_{\underline{H}}$  on  $\mathbb{A}_{\underline{H}}$ . Recall that for any  $(a, h_a) \in \mathbb{A}_{\underline{H}}$ ,  $\phi_{\underline{H}}U_{\underline{H}}(a, h_a) = (H(a), m_a)$  and  $\widehat{\overline{H}}(a, h_a) = (H(a), H(h_a) \cdot \lambda_a)$ .

As pointed out in [21], for any object  $b$  of  $\mathbb{A}$ ,  $K_{\underline{H}}(b) = (H(b), \alpha_{H(b)})$  for some  $\alpha : H(b) \rightarrow HH(b)$ , thus inducing a natural transformation

$$\alpha_{K_{\underline{H}}} : \phi_{\underline{H}} \rightarrow \widehat{\overline{H}}\phi_{\underline{H}},$$

whose component at  $b \in \mathbb{A}$  is  $\alpha_{H(b)}$ , we may choose it to be just  $\delta_b$ , and we have a morphism of comonads

$$t_{K_{\underline{H}}} : \phi_{\underline{H}}U_{\underline{H}} \xrightarrow{\alpha_{K_{\underline{H}}}U_{\underline{H}}} \widehat{\overline{H}}\phi_{\underline{H}}U_{\underline{H}} \xrightarrow{\widehat{\overline{H}}\varepsilon_{\underline{H}}} \widehat{\overline{H}},$$

where  $\varepsilon_{\underline{H}}$  is the counit of the adjunction  $\phi_{\underline{H}} \dashv U_{\underline{H}}$ , and since  $(\varepsilon_{\underline{H}})_{(a, h_a)} = h_a$ , we see that for all  $(a, h_a) \in \mathbb{A}_{\underline{H}}$ ,  $(t_{K_{\underline{H}}})_{(a, h_a)}$  is the composite

$$(4.5) \quad H(a) \xrightarrow{\delta_a} HH(a) \xrightarrow{H(h_a)} H(a).$$

**(ii) The functor  $\phi^{\overline{H}}$ .** The cofree  $\overline{H}$ -comodule functor  $\phi^{\overline{H}}$  has the forgetful functor  $U^{\overline{H}} : \mathbb{A}^{\overline{H}} \rightarrow \mathbb{A}$  as a left adjoint. The unit  $\eta : 1 \rightarrow \phi^{\overline{H}} U^{\overline{H}}$  and counit  $\sigma : U^{\overline{H}} \phi^{\overline{H}} \rightarrow 1$  of the adjunction  $U^{\overline{H}} \dashv \phi^{\overline{H}}$  are given by the formulas:

$$\eta_{(a, \theta_a)} = \theta_a : (a, \theta_a) \rightarrow \phi^{\overline{H}} U^{\overline{H}}(a, \theta_a) = (H(a), \delta_a)$$

and

$$\sigma_a = \varepsilon_a : H(a) = U^{\overline{H}} \phi^{\overline{H}}(a) \rightarrow a.$$

Since  $\varepsilon$  is a split epimorphism, it follows from Corollary 3.17 of [20] that, when  $\mathbb{A}$  is Cauchy complete, the functor  $\phi^{\overline{H}}$  is monadic.

Since  $K_{\overline{H}}(a) = ((H(a), \delta_a), m_a)$ , it is easy to see that the  $a$ -component of

$$\alpha_{K_{\overline{H}}} : \widehat{\underline{H}} K_{\overline{H}} \rightarrow K_{\overline{H}}$$

is just the morphism  $m_a : HH(a) \rightarrow H(a)$ , and we have a monad morphism

$$t_{K_{\overline{H}}} : \widehat{\underline{H}} \xrightarrow{\widehat{\underline{H}}\eta} \underline{H} \phi^{\overline{H}} U^{\overline{H}} \xrightarrow{\alpha_{K_{\overline{H}}} U^{\overline{H}}} \phi^{\overline{H}} U^{\overline{H}}.$$

It follows that for any  $(a, \theta_a) \in \mathbb{A}^{\overline{H}}$ ,  $(t_{K_{\overline{H}}})_{(a, \theta_a)}$  is the composite

$$(4.6) \quad H(a) \xrightarrow{H(\theta_a)} HH(a) \xrightarrow{m_a} H(a).$$

**4.4. The comparison functor as an equivalence.** Let  $\mathbb{A}$  be a Cauchy complete category. For a bimonad  $\mathbf{H} = (\underline{H} = (H, m, e), \overline{H} = (H, \delta, \varepsilon), \lambda)$ , the following are equivalent:

- (a)  $K_H : \mathbb{A} \rightarrow \mathbb{A}_{\underline{H}}^{\overline{H}}(\lambda)$ ,  $a \rightarrow (H(a), \delta_a, m_a)$ , is an equivalence of categories;
- (b)  $t_{K_{\underline{H}}} : \phi_{\underline{H}} U_{\underline{H}} \rightarrow \widehat{\underline{H}}$  is an isomorphism of comonads;
- (c) for any  $(a, h_a) \in \mathbb{A}_{\underline{H}}$ , the composite  $H(h_a) \cdot \delta_a$  is an isomorphism;
- (d)  $t_{K_{\overline{H}}} : \widehat{\underline{H}} \rightarrow \phi^{\overline{H}} U^{\overline{H}}$  is an isomorphism of monads;
- (e) for any  $(a, \theta_a) \in \mathbb{A}^{\overline{H}}$ , the composite  $m_a \cdot H(\theta_a)$  is an isomorphism.

**Proof.** We may identify the functors  $K_{\underline{H}}$ ,  $K_{\overline{H}}$  and  $K_H$ .

(a) $\Leftrightarrow$ (b) Since  $\mathbb{A}$  is Cauchy complete and since the unit  $\eta_{\underline{H}} : 1 \rightarrow U_{\underline{H}} \phi_{\underline{H}}$  of the adjunction  $\phi_{\underline{H}} \dashv U_{\underline{H}}$  is a split monomorphism, the functor  $\phi_{\underline{H}}$  is comonadic by the dual of [19, Theorem 6]. Now, by [21, Theorem 4.4.],  $K_{\underline{H}}$  is an equivalence if and only if  $t_{K_{\underline{H}}}$  is an isomorphism.

(b) $\Leftrightarrow$ (c) and (d) $\Leftrightarrow$ (e). By 4.3, the morphisms in (b) come out as the morphisms in (c), and the morphisms in (d) are just those in (e).

(a) $\Leftrightarrow$ (d) Since  $\varepsilon$  is a split epimorphism, it follows from [20, Corollary 3.17] that (since  $\mathbb{A}$  is Cauchy complete) the functor  $\phi^{\overline{H}}$  is monadic and hence  $K$  is an equivalence by 3.7.  $\square$

## 5. ANTIPODE

We consider a bimonad  $\mathbf{H} = (H, m, e, \delta, \varepsilon, \lambda)$  on any category  $\mathbb{A}$ .

5.1. **Canonical maps.** Define the composites

$$(5.1) \quad \begin{aligned} \gamma : HH &\xrightarrow{\delta H} HHH \xrightarrow{Hm} HH, \\ \gamma' : HH &\xrightarrow{H\delta} HHH \xrightarrow{mH} HH. \end{aligned}$$

In the diagram

$$\begin{array}{ccccc} HHH & \xrightarrow{\delta HH} & HHHH & \xrightarrow{HmH} & HHH \\ \downarrow Hm & & \downarrow HHm & & \downarrow Hm \\ HH & \xrightarrow{\delta H} & HHH & \xrightarrow{Hm} & HH, \end{array}$$

the left square commutes by naturality of  $\delta$ , while the right square commutes by associativity of  $m$ . From this we see that  $\gamma$  is left  $\underline{H}$ -linear as a morphism from  $(HH, Hm)$  to itself. A similar diagram shows that  $\gamma'$  is right  $\underline{H}$ -linear as a morphism from  $(HH, mH)$  to itself. Moreover, in the diagram

$$\begin{array}{ccccc} H & \xrightarrow{He} & HH & \xrightarrow{\delta H} & HHH \\ \downarrow \delta & & \nearrow HHe & & \downarrow Hm \\ HH & \xlongequal{\quad\quad\quad} & & & HH \end{array}$$

the top triangle commutes by functoriality of composition, while the bottom triangle commutes because  $m \cdot He = 1$ . Drawing a similar diagram for  $H\delta$  and  $mH$ , we obtain

$$(5.2) \quad \gamma \cdot He = \delta, \quad \gamma' \cdot eH = \delta.$$

5.2. **Definition.** A natural transformation  $S : H \rightarrow H$  is said to be

- a *left antipode* if  $m \cdot (SH) \cdot \delta = e \cdot \varepsilon$ ;
- a *right antipode* if  $m \cdot (HS) \cdot \delta = e \cdot \varepsilon$ ;
- an *antipode* if it is a left and a right antipode.

A bimonad  $\mathbf{H}$  is said to be a *Hopf monad* provided it has an antipode.

Following the pattern of the proof of [8, 15.2] we obtain:

5.3. **Proposition.** We refer to the notation in 5.1.

- (1) If  $\gamma$  has an  $\overline{H}$ -linear left inverse, then  $\mathbf{H}$  has a left antipode.
- (2) If  $\gamma'$  has an  $\overline{H}$ -linear left inverse, then  $\mathbf{H}$  has a right antipode.

**Proof.** (1) Suppose there exists an  $\mathbf{H}$ -linear morphism  $\beta : HH \rightarrow HH$  with  $\beta \cdot \gamma = 1$ . Consider the composite

$$S : H \xrightarrow{He} HH \xrightarrow{\beta} HH \xrightarrow{\varepsilon H} H.$$

We claim that  $S$  is a left antipode of  $\mathbf{H}$ . Indeed, in the diagram

$$\begin{array}{ccccccc}
 H & \xrightarrow{\delta} & HH & \xrightarrow{HeH} & HHH & \xrightarrow{\beta H} & HHH & \xrightarrow{\varepsilon HH} & HH \\
 & & \searrow & \downarrow Hm & & (1) & \downarrow Hm & (2) & \downarrow m \\
 & & & HH & \xrightarrow{\beta} & HH & \xrightarrow{\varepsilon H} & H,
 \end{array}$$

the *triangle* commutes since  $e$  is the unit for the monad  $\underline{H}$ , *rectangle* (1) commutes by  $\underline{H}$ -linearity of  $\beta$ , and *rectangle* (2) commutes by naturality of  $\varepsilon$ . Thus

$$m \cdot SH \cdot \delta = m \cdot \varepsilon HH \cdot \beta H \cdot HeH \cdot \delta = \varepsilon H \cdot \beta \cdot \delta,$$

and using (5.2), we have

$$\varepsilon H \cdot \beta \cdot \delta = \varepsilon H \cdot \beta \cdot \gamma \cdot He = \varepsilon H \cdot He = e \cdot \varepsilon.$$

Therefore  $S$  is a left antipode of  $\mathbf{H}$ .

(2) Denoting the left inverse of  $\gamma'$  by  $\beta'$ , it is shown along the same lines that  $S' = H\varepsilon \cdot \beta' \cdot eH$  is a right antipode.  $\square$

**5.4. Lemma.** *Suppose that  $\gamma$  is an epimorphism. If  $f, g : H \rightarrow H$  are two natural transformations such that*

$$m \cdot fH \cdot \delta = m \cdot gH \cdot \delta \quad \text{or} \quad m \cdot Hf \cdot \delta = m \cdot Hg \cdot \delta,$$

*then  $f = g$ .*

**Proof.** Assume  $m \cdot fH \cdot \delta = m \cdot gH \cdot \delta$ . Since  $\gamma \cdot He = \delta$  by (5.2), we have

$$m \cdot fH \cdot \gamma \cdot He = m \cdot gH \cdot \gamma \cdot He,$$

and, since  $\gamma$  is also  $\underline{H}$ -linear, it follows by Lemma 3.2 that

$$m \cdot fH \cdot \gamma = m \cdot gH \cdot \gamma.$$

But  $\gamma$  is an epimorphism by our assumption, thus

$$m \cdot fH = m \cdot gH.$$

By naturality of  $e : 1 \rightarrow H$ , we have the commutative diagrams

$$\begin{array}{ccc}
 H & \xrightarrow{f} & H \\
 He \downarrow & & \downarrow He \\
 HH & \xrightarrow{fH} & HH,
 \end{array}
 \quad
 \begin{array}{ccc}
 H & \xrightarrow{g} & H \\
 He \downarrow & & \downarrow He \\
 HH & \xrightarrow{gH} & HH.
 \end{array}$$

Thus, since  $m \cdot He = 1$ ,

$$f = m \cdot He \cdot f = m \cdot fH \cdot He = m \cdot gH \cdot He = m \cdot He \cdot g = g.$$

If  $m \cdot Hf \cdot \delta = m \cdot Hg \cdot \delta$  similar arguments apply.  $\square$

**5.5. Characterising Hopf monads.** *Let  $\mathbf{H} = (H, m, e, \delta, \varepsilon, \lambda)$  be a bimonad.*

(1) *The following are equivalent:*

- (a)  $\gamma = Hm \cdot \delta H : HH \rightarrow HH$  is an isomorphism;
- (b)  $\gamma' = mH \cdot H\delta : HH \rightarrow HH$  is an isomorphism;
- (c)  $\mathbf{H}$  has an antipode.



(2) If  $\mathbf{H}$  has an antipode and  $\mathbb{A}$  admits equalisers, then the comparison functor (see 4.3)

$$K_{\underline{H}} : \mathbb{A} \rightarrow \mathbb{A}_{\underline{H}}^{\overline{H}}(\lambda)$$

makes  $\mathbb{A}$  (isomorphic to) a coreflective subcategory of the category  $\mathbb{A}_{\underline{H}}^{\overline{H}}(\lambda)$ .

**Proof.** (1) (c) $\Rightarrow$ (a) The proof for [21, Proposition 6.10] applies almost literally.

(a) $\Rightarrow$ (c) Write  $\beta : HH \rightarrow HH$  for the inverse of  $\gamma$ . Since  $\gamma$  is  $\underline{H}$ -linear, it follows that  $\beta$  also is  $\underline{H}$ -linear. Then, by Proposition 5.3,  $S = \varepsilon H \cdot \beta \cdot He$  is a left antipode of  $\mathbf{H}$ . We show that  $S$  is also a right antipode of  $\mathbf{H}$ . In the diagram

$$\begin{array}{ccccccccc}
 H & \xrightarrow{\delta} & HH & \xrightarrow{\delta H} & HHH & \xrightarrow{HSH} & HHH & \xrightarrow{mH} & HH \\
 & \searrow \delta & & \nearrow H\delta & & & \downarrow Hm & & \downarrow m \\
 & & HH & \xrightarrow{H\varepsilon} & H & \xrightarrow{He} & HH & \xrightarrow{m} & H
 \end{array}$$

(1) (2) (3)

- (1) commutes by coassociativity of  $\delta$ ;
- (2) commutes because  $S$  is a left antipode of  $\mathbf{H}$ ;
- (3) commutes by associativity of  $m$ .

Since  $m \cdot He = 1 = m \cdot eH$  and  $H\varepsilon \cdot \delta = 1 = \varepsilon H \cdot \delta$ , it follows that

$$\begin{aligned}
 m \cdot (m \cdot HS \cdot \delta)H \cdot \delta &= m \cdot mH \cdot HSH \cdot \delta H \cdot \delta = m \cdot He \cdot H\varepsilon \cdot \delta \\
 &= m \cdot eH \cdot \varepsilon H \cdot \delta = m \cdot ((e \cdot \varepsilon)H) \cdot \delta.
 \end{aligned}$$

$\gamma$  being an epimorphism, Lemma 5.4 implies  $m \cdot HS \cdot \delta = e \cdot \varepsilon$ , proving that  $S$  is also a right antipode of  $\mathbf{H}$ .

(b) $\Leftrightarrow$ (c) can be shown in a similar way.

(2) Since

- to say that  $\gamma$  is an isomorphism is to say that  $(t_{K_{\underline{H}}})_{(H(a), m_a)}$  is an isomorphism for all  $a \in \mathbb{A}$ ;
- $(H(a), m_a) = \phi_{\underline{H}}(a)$ ;
- the unit  $\eta_{\underline{H}} : 1 \rightarrow \phi_{\underline{H}}U_{\underline{H}}$  of the adjunction  $\phi_{\underline{H}} \dashv U_{\underline{H}}$  is just  $e : 1 \rightarrow H$ , which is a split monomorphism,

we can apply Corollary 3.15 to get the desired result.  $\square$

Combining 5.5 and 4.4, we get:

**5.6. Antipode and equivalence.** Let  $\mathbf{H} = (H, m, e, \delta, \varepsilon, \lambda)$  be a bimonad on a category  $\mathbb{A}$  and assume that  $\mathbb{A}$  admits colimits or limits and  $H$  preserves them. Then the following are equivalent:

- (a)  $\mathbf{H}$  has an antipode;
- (b)  $\gamma = Hm \cdot \delta H : HH \rightarrow HH$  is an isomorphism;
- (c)  $\gamma' = mH \cdot H\delta : HH \rightarrow HH$  is an isomorphism;
- (d)  $K_H : \mathbb{A} \rightarrow \mathbb{A}_{\underline{H}}^{\overline{H}}(\lambda)$ ,  $a \rightarrow (H(a), \delta_a, m_a)$ , is an equivalence.

**Proof.** (a) $\Leftrightarrow$ (b) $\Leftrightarrow$ (c) (in any category) is shown in 5.5.

(b) $\Leftrightarrow$ (d) Since  $H$  preserves colimits, the category  $\mathbb{A}_{\underline{H}}$  admits colimits and the functor  $U_{\underline{H}} : \mathbb{A}_{\underline{H}} \rightarrow \mathbb{A}$  creates them (see, for example, [24]). Thus

- the functor  $\phi_{\underline{H}}U_{\underline{H}}$  preserves colimits;

- any functor  $L : \mathbb{B} \rightarrow \mathbb{A}_{\underline{H}}$  preserves colimits if and only if the composite  $U_{\underline{H}}L$  does; so, in particular, the functor  $\widehat{\underline{H}}$  preserves colimits, since  $U_{\underline{H}}\widehat{\underline{H}} = HU_{\underline{H}}$  and since the functor  $HU_{\underline{H}}$ , being the composite of two colimit-preserving functors, is colimit-preserving.

The full subcategory of  $\mathbb{A}_{\underline{H}}$  given by the free  $\underline{H}$ -modules is dense and since the functors  $\phi_{\underline{H}}U_{\underline{H}}$  and  $\widehat{\underline{H}}$  both preserve colimits, it follows from [24, Theorem 17.2.7] that the natural transformation (see 4.4)

$$t_{K_{\underline{H}}} : \phi_{\underline{H}}U_{\underline{H}} \rightarrow \widehat{\underline{H}}$$

is an isomorphism if and only if its restriction to the free  $\underline{H}$ -modules is so; i.e. if  $(t_{K_{\underline{H}}})_{\phi_{\underline{H}}(a)}$  is an isomorphism for all  $a \in \mathbb{A}$ . But since  $\phi_{\underline{H}}(a) = (H(a), m_a)$ ,  $t_{K_{\underline{H}}}$  is an isomorphism if and only if the composite

$$HH(a) \xrightarrow{\delta_{H(a)}} HHH(a) \xrightarrow{H(m_a)} HH(a)$$

is an isomorphism for all  $a \in \mathbb{A}$ , that is, the isomorphism

$$\gamma : HH \xrightarrow{\delta H} HHH \xrightarrow{Hm} HH.$$

(c) $\Leftrightarrow$ (d) Since the functor  $H$  preserves limits, the category  $\mathbb{A}^{\overline{H}}$  admits and the functor  $U^{\overline{H}}$  creates limits. Since  $\phi^{\overline{H}}$ , being right adjoint, preserves limits, the functor  $\phi^{\overline{H}}U^{\overline{H}}$  also preserves limits. Moreover, since the monad  $\widehat{\underline{H}}$  is a lifting of the monad  $\underline{H}$  along the functor  $U^{\overline{H}}$ ,  $U^{\overline{H}}\widehat{\underline{H}} = HU^{\overline{H}}$ , implying that the functor  $\widehat{\underline{H}}$  also preserves limits. Now, since the full subcategory of  $\mathbb{A}^{\overline{H}}$  spanned by cofree  $\overline{H}$ -comodules is codense, it follows from the dual of [24, Theorem 17.2.7] that the natural transformation  $t_{K_{\overline{H}}}$  (see 4.4) is an isomorphism if and only if its restriction to free  $\overline{H}$ -comodules is so. But for any  $a \in \mathbb{A}$ ,  $(t_{K_{\overline{H}}})_{(H(a), \delta_a)} = m_{H(a)} \cdot H(\delta_a)$ . Thus  $t_{K_{\overline{H}}}$  is an isomorphism if and only if the composite  $\gamma'$  is an isomorphism.  $\square$

## 6. LOCAL PREBRAIDINGS FOR HOPF MONADS

For any category  $\mathbb{A}$  we now fix a system  $\mathbf{H} = (H, m, e, \delta, \varepsilon)$  consisting of an endofunctor  $H : \mathbb{A} \rightarrow \mathbb{A}$  and natural transformations  $m : HH \rightarrow H$ ,  $e : 1 \rightarrow H$ ,  $\delta : H \rightarrow HH$  and  $\varepsilon : H \rightarrow 1$  such that the triple  $\underline{H} = (H, m, e)$  is a monad and the triple  $\overline{H} = (H, \delta, \varepsilon)$  is a comonad on  $\mathbb{A}$ .

**6.1. Double entwining.** A natural transformation  $\tau : HH \rightarrow HH$  is called a *double entwining* if

- (i)  $\tau$  is a mixed distributive law from the monad  $\underline{H}$  to the comonad  $\overline{H}$ ;
- (ii)  $\tau$  is a mixed distributive law from the comonad  $\overline{H}$  to the monad  $\underline{H}$ .

These conditions are obviously equivalent to

- (iii)  $\tau$  is a monad distributive law for the monad  $\underline{H}$ ;
- (iv)  $\tau$  is a comonad distributive law for the comonad  $\overline{H}$ .

Explicitly (i) encodes the identities

$$(6.1) \quad He = \tau \cdot eH$$

$$(6.2) \quad H\varepsilon = \varepsilon H \cdot \tau$$

$$(6.3) \quad \delta H \cdot \tau = H\tau \cdot \tau H \cdot H\delta$$

$$(6.4) \quad \tau \cdot mH = Hm \cdot \tau H \cdot H\tau,$$

and (ii) is equivalent to the identities

$$(6.5) \quad eH = \tau \cdot He$$

$$(6.6) \quad \varepsilon H = H\varepsilon \cdot \tau$$

$$(6.7) \quad H\delta \cdot \tau = \tau H \cdot H\tau \cdot \delta H$$

$$(6.8) \quad \tau \cdot Hm = mH \cdot H\tau \cdot \tau H$$

**6.2.  $\tau$ -bimonad.** Let  $\tau : HH \rightarrow HH$  be a double entwining. Then  $\mathbf{H}$  is called a  $\tau$ -bimonad provided the diagram

$$(6.9) \quad \begin{array}{ccccc} HH & \xrightarrow{m} & H & \xrightarrow{\delta} & HH \\ \delta\delta \downarrow & & & & \uparrow mm \\ HHH & \xrightarrow{H\tau H} & HHH & & HHH \end{array}$$

is commutative, that is

$$\delta \cdot m = mm \cdot H\tau H \cdot \delta\delta = Hm \cdot mHH \cdot H\tau H \cdot HH\delta \cdot \delta H,$$

and also the following diagrams commute

$$(6.10) \quad \begin{array}{ccc} HH \xrightarrow{H\varepsilon} H & 1 \xrightarrow{e} H & 1 \xrightarrow{e} H \\ m \downarrow & e \downarrow & \searrow = \\ H \xrightarrow{\varepsilon} 1, & H \xrightarrow{eH} HH, & \downarrow \varepsilon \\ & & 1. \end{array}$$

**6.3. Proposition.** Let  $\mathbf{H}$  be a  $\tau$ -bimonad. Then the composite

$$\tilde{\tau} : HH \xrightarrow{\delta H} HHH \xrightarrow{H\tau} HHH \xrightarrow{mH} HH$$

is a mixed distributive law from the monad  $\underline{H}$  to the comonad  $\overline{H}$ . Thus  $\mathbf{H}$  is a bimonad (as in 4.1) with mixed distributive law  $\tilde{\tau}$ .

**Proof.** We have to show that  $\tilde{\tau}$  satisfies

$$(6.11) \quad He = \tilde{\tau} \cdot eH$$

$$(6.12) \quad H\varepsilon = \varepsilon H \cdot \tilde{\tau}$$

$$(6.13) \quad \delta H \cdot \tilde{\tau} = H\tilde{\tau} \cdot \tilde{\tau} H \cdot H\delta$$

$$(6.14) \quad \tilde{\tau} \cdot mH = Hm \cdot \tilde{\tau} H \cdot H\tilde{\tau}$$

Consider the diagram

$$\begin{array}{ccccccc} H & \xrightarrow{eH} & HH & \xrightarrow{\tau} & HH & & \\ eH \downarrow & & eHH \downarrow & & eH \downarrow & \searrow & \\ HH & \xrightarrow{\delta H} & HHH & \xrightarrow{H\tau} & HHH & \xrightarrow{mH} & HH, \end{array}$$

(1)                      (2)

which is commutative since *square* (1) commutes by (6.10); *square* (2) commutes by functoriality of composition; the *triangle* commutes since  $e$  is the identity of the monad  $\underline{H}$ . Thus  $\tilde{\tau} \cdot eH = mH \cdot H\tau \cdot \delta H \cdot eH = \tau \cdot eH$ , and (6.1) implies  $\tilde{\tau} \cdot eH = He$ , showing (6.11).

Consider now the diagram

$$\begin{array}{ccccccc}
 HH & \xrightarrow{\delta H} & HHH & \xrightarrow{H\tau} & HHH & \xrightarrow{mH} & HH \\
 & & \downarrow \varepsilon HH & \searrow HH\varepsilon & \downarrow H\varepsilon H & \downarrow \varepsilon H & \\
 & & HH & & HH & \xrightarrow{\varepsilon H} & H \\
 & & & \nearrow H\varepsilon & & & \\
 & & HH & & & & 
 \end{array}
 \quad (1)$$

in which *square* (1) commutes because  $\varepsilon$  is a morphism of monads and thus  $\varepsilon \cdot m = \varepsilon \cdot H\varepsilon$ ; the *triangle* commutes because of (6.2), *diagram* (2) commutes because of functoriality of composition.

Thus  $\varepsilon H \cdot \tilde{\tau} = \varepsilon H \cdot mH \cdot H\tau \cdot \delta H = H\varepsilon \cdot \varepsilon HH \cdot \delta H = H\varepsilon$ , showing (6.12).

Constructing suitable commutative diagram we can show

$$\begin{aligned}
 \tilde{\tau} \cdot mH &= mH \cdot H\tau \cdot \delta H \cdot mH \\
 &= mH \cdot HHm \cdot HmHH \cdot HH\tau H \cdot H\tau HH \cdot HHH\tau \cdot \delta\delta H, \\
 Hm \cdot \tilde{\tau} H \cdot H\tilde{\tau} &= Hm \cdot mHH \cdot H\tau H \cdot \delta HH \cdot HmH \cdot HH\tau \cdot H\delta H \\
 &= mH \cdot HHm \cdot HmHH \cdot HH\tau H \cdot H\tau HH \cdot HHH\tau \cdot \delta\delta H.
 \end{aligned}$$

Comparing this two identities we get the condition (6.14).

To show that (6.13) also holds, consider the diagram

$$\begin{array}{ccccccc}
 HHH & \xrightarrow{\delta HH} & HHHH & \xrightarrow{H\tau H} & HHHH & \xrightarrow{mHH} & HHH \\
 & \searrow \delta HH & \downarrow H\delta HH & \searrow HH\delta H & \downarrow HH\delta H & \downarrow H\delta H & \\
 & & HHHHH & \xrightarrow{mHHH} & HHHH & & \\
 & \nearrow \delta HHH & \downarrow HH\tau H & \nearrow H\tau HH & \downarrow HHH\tau & \downarrow HH\tau & \\
 HH & \xrightarrow{\delta\delta} & HHHH & & HHHH & \xrightarrow{mHHH} & HHHH \\
 & & & \searrow mmH & & \downarrow HmH & \\
 & & & & & & HHH,
 \end{array}
 \quad (1) \quad (2) \quad (3) \quad (4)$$

in which the *triangles* and *diagrams* (1) and (3) commute by functoriality of composition; *diagram* (2) commutes by (6.7); *diagram* (4) commutes by naturality of  $m$ .

Finally we construct the diagram

$$\begin{array}{ccccccc}
 HH & \xrightarrow{\delta H} & HHH & \xrightarrow{H\tau} & HHH & \xrightarrow{mH} & HH \\
 \downarrow \delta\delta & \nearrow HH\delta & & \downarrow H\delta H & & \downarrow \delta H & \\
 HHHH & \xrightarrow{H\tau H} & HHHH & \xrightarrow{HH\tau} & HHHH & & HHH \\
 \downarrow \delta HHH & & \downarrow \delta HH & & \downarrow \delta HHH & & \uparrow mmH \\
 HHHHH & \xrightarrow{HH\tau H} & HHHHH & \xrightarrow{HHH\tau} & HHHHH & \xrightarrow{H\tau HH} & HHHHH \\
 & & \searrow H\tau HH & & \nearrow HHH\tau & & \\
 & & & & HHHHH & & 
 \end{array}
 \quad (1) \quad (2) \quad (3) \quad (4) \quad (5)$$

in which *diagram* (1) commutes by (6.3); *diagram* (2) commutes by (6.9) because  $\delta H H H \cdot H \delta H = \delta \delta H$ ; the *triangle* and *diagrams* (3), (4) and (5) commute by functoriality of composition.

It now follows from the commutativity of these diagrams that

$$\begin{aligned} \delta H \cdot \tilde{\tau} &= \delta H \cdot m H \cdot H \tau \cdot \delta H \\ &= m m H \cdot H H H \tau \cdot H \tau H H \cdot H H \tau H \cdot \delta H H H \cdot \delta \delta \\ &= (H m H \cdot H H \tau \cdot H \delta H) \cdot (m H H \cdot H \tau H \cdot \delta H H) \cdot H \delta \\ &= H \tilde{\tau} \cdot \tilde{\tau} H \cdot H \delta. \end{aligned}$$

Therefore  $\tilde{\tau}$  satisfies the conditions (6.11)-(6.14) and hence is a mixed distributive law from the monad  $\underline{H}$  to the comonad  $\overline{H}$ .  $\square$

**6.4. Corollary.** *In the situation of the previous proposition, if  $(a, \theta_a) \in \mathbb{A}^{\overline{H}}$ , then  $(H(a), \theta_{H(a)}) \in \mathbb{A}^{\overline{H}}$ , where  $\theta_{H(a)}$  is the composite*

$$H(a) \xrightarrow{H(\theta_a)} H H(a) \xrightarrow{\delta_{H(a)}} H H H(a) \xrightarrow{H \tau_a} H H H(a) \xrightarrow{m_{H(a)}} H H(a) .$$

**Proof.** Write  $\widehat{H}$  for the monad on the category  $\mathbb{A}^{\overline{H}}$  that is the lifting of  $\underline{H}$  corresponding to the mixed distributive law  $\tilde{\tau}$ . Since  $\theta_{H(a)} = \tilde{\tau}_a \cdot H(\theta_a)$ , it follows that  $(H(a), \theta_{H(a)}) = \widehat{H}(a, \theta_a)$ , and thus  $(H(a), \theta_{H(a)})$  is an object of the category  $\mathbb{A}^{\overline{H}}$ .  $\square$

**6.5.  $\tau$ -Bimodules.** Given the conditions of Proposition 6.3, we have the commutative diagram (see (4.1))

$$\begin{array}{ccccc} H H & \xrightarrow{m} & H & \xrightarrow{\delta} & H H \\ H \delta \downarrow & & & & \uparrow H m \\ H H H & \xrightarrow{\tilde{\tau} H} & H H H, & & \end{array}$$

and thus  $H$  is a bimonad by the entwining  $\tilde{\tau}$  and the mixed bimodules are objects  $a$  in  $\mathbb{A}$  with a module structure  $h_a : H(a) \rightarrow a$  and a comodule structure  $\theta_a : a \rightarrow H(a)$  with a commutative diagram

$$\begin{array}{ccccc} H(a) & \xrightarrow{h_a} & a & \xrightarrow{\theta_a} & H(a) \\ H(\theta_a) \downarrow & & & & \uparrow H(h_a) \\ H H(a) & \xrightarrow{\tilde{\tau}_a} & H H(a). & & \end{array}$$

By definition of  $\tilde{\tau}$ , commutativity of this diagram is equivalent to the commutativity of

$$(6.15) \quad \begin{array}{ccccc} & H(a) & \xrightarrow{h_a} & a & \xrightarrow{\theta_a} & H(a) \\ & \swarrow H(\theta_a) & & & & \nwarrow H(h_a) \\ H H(a) & & & & & H H(a) \\ & \searrow \delta_{H(a)} & & & & \nearrow m_{H(a)} \\ & & H H H(a) & \xrightarrow{H(\tau_a)} & H H H(a) & \end{array} .$$

A morphism  $f : (a, h_a, \theta_a) \rightarrow (a', h_{a'}, \theta_{a'})$  is a morphism  $f : a \rightarrow a'$  such that  $f \in \mathbb{A}^{\overline{H}}$  and  $f \in \mathbb{A}_{\underline{H}}$ .

We denote the category  $\mathbb{A}_{\underline{H}}^{\overline{H}}(\tilde{\tau})$  by  $\mathbb{A}_H^H$ .

**6.6. Antipode of a  $\tau$ -bimonad.** Let  $\mathbf{H} = (H, m, e, \delta, \varepsilon)$  be a  $\tau$ -bimonad with an antipode  $S$  where  $\tau : HH \rightarrow HH$  is a double entwining. Then

$$(6.16) \quad S \cdot m = m \cdot SS \cdot \tau \quad \text{and} \quad \delta \cdot S = \tau \cdot SS \cdot \delta.$$

If  $\tau \cdot HS = SH \cdot \tau$  and  $\tau \cdot SH = HS \cdot \tau$ , then  $S : H \rightarrow H$  is a monad as well as a comonad morphism.

**Proof.** Since  $(HH, H\tau H \cdot \delta, \varepsilon\varepsilon)$  is a comonad and  $(H, m, e)$  is a monad, the collection  $\text{Nat}(HH, H)$  of all natural transformations from  $HH$  to  $H$  forms a semigroup with unit  $e \cdot \varepsilon\varepsilon$  and with product

$$f * g : HH \xrightarrow{\delta\delta} HHHH \xrightarrow{H\tau H} HHHH \xrightarrow{fg} HH \xrightarrow{m} H.$$

Consider now the diagram

$$\begin{array}{ccccc}
 & HH & \xrightarrow{\delta\delta} & HHHH & \xrightarrow{H\tau H} & HHHH \\
 & \downarrow m & & & & \downarrow mHH \\
 H & \xleftarrow{H\varepsilon} & H & \xrightarrow{\delta} & HH & \xleftarrow{Hm} & HHH \\
 & \downarrow \varepsilon & & & & \downarrow SH \\
 & I & \xrightarrow{e} & H & \xleftarrow{m} & HH \\
 & & & & & \downarrow Hm \\
 & & & & & HH
 \end{array}$$

(1) (2) (3) (4)

in which the diagrams (1), (2) and (3) commute because  $H$  is a bimonad, while diagram (4) commutes by naturality. It follows that

$$m \cdot Hm \cdot SHH \cdot mHH \cdot H\tau H \cdot \delta\delta = e \cdot \varepsilon \cdot H\varepsilon = \varepsilon\varepsilon \cdot e.$$

Thus  $S \cdot m = m^{-1}$  in  $\text{Nat}(HH, H)$ . Furthermore, by (a somewhat tedious) computation we can show

$$m \cdot Hm \cdot HHS \cdot HSH \cdot H\tau \cdot mHH \cdot H\tau H \cdot \delta\delta = e \cdot \varepsilon \cdot H\varepsilon = e \cdot \varepsilon\varepsilon.$$

This shows that  $m \cdot SS \cdot \tau = m^{-1}$  in  $\text{Nat}(HH, H)$ . Thus  $m \cdot SS \cdot \tau = S \cdot m$ .

To prove the formula for the coproduct consider  $\text{Nat}(H, HH)$  as a monoid with unit  $ee \cdot \varepsilon$  and the convolution product for  $f, g \in \text{Nat}(H, HH)$  given by

$$f * g : H \xrightarrow{\delta} HH \xrightarrow{fH} HHH \xrightarrow{HHg} HHHH \xrightarrow{mm} HH.$$

By computation we get

$$\begin{aligned}
 (\delta \cdot S) * \delta &= eH \cdot e \cdot \varepsilon = ee \cdot \varepsilon, \\
 \delta * (\tau \cdot SS \cdot \delta) &= He \cdot e \cdot \varepsilon = ee \cdot \varepsilon.
 \end{aligned}$$

Thus  $(\delta \cdot S) * \delta = 1$  and  $\delta * (\tau \cdot SS \cdot \delta) = 1$ , and hence  $\delta \cdot S = \tau \cdot SS \cdot \delta$ .

Now assume  $\tau \cdot HS = SH \cdot \tau$  and  $\tau \cdot SH = HS \cdot \tau$ . Then we have

$$SS \cdot \tau = SH \cdot HS \cdot \tau = SH \cdot \tau \cdot SH = \tau \cdot HS \cdot SH = \tau \cdot SS, \text{ thus}$$

$$S \cdot m = m \cdot SS \cdot \tau = m \cdot \tau \cdot SS = m' \cdot SS.$$

Moreover, since  $m \cdot He = 1$ , we have

$$S \cdot e = m \cdot He \cdot S \cdot e \stackrel{\text{nat}}{=} m \cdot SH \cdot He \cdot e \stackrel{(6.10)}{=} m \cdot SH \cdot \delta \cdot e \stackrel{\text{antip.}}{=} e \cdot \varepsilon \cdot e \stackrel{(6.10)}{=} e.$$

Hence  $S$  is a monad morphism from  $(H, m, e)$  to  $(H, m \cdot \tau, e)$ .

For the coproduct,  $SS \cdot \tau = \tau \cdot SS$  implies

$$\delta \cdot S = \tau \cdot SS \cdot \delta = SS \cdot \tau \cdot \delta = SS \cdot \delta'.$$

Furthermore,

$$\varepsilon \cdot S = \varepsilon \cdot S \cdot H\varepsilon \cdot \delta \stackrel{\text{nat}}{=} \varepsilon \cdot H\varepsilon \cdot SH \cdot \delta \stackrel{(6.10)}{=} \varepsilon \cdot m \cdot SH \cdot \delta \stackrel{\text{antip.}}{=} \varepsilon \cdot e \cdot \varepsilon \stackrel{(6.10)}{=} \varepsilon.$$

This shows that  $S$  is a comonad morphism from  $(H, \delta, \varepsilon)$  to  $(H, \tau \cdot \delta, \varepsilon)$ .  $\square$

It is readily checked that for a bimonad  $H$ , the composite  $HH$  is again a comonad as well as a monad. However, the compatibility between these two structures needs an additional property of the double entwining  $\tau$ . This will also help to construct a bimonad "opposite" to  $H$ .

**6.7. Local prebraiding.** Let  $\tau : HH \rightarrow HH$  be a natural transformation.  $\tau$  is said to satisfy the *Yang-Baxter equation (YB)* if it induces commutativity of the diagram

$$\begin{array}{ccccc} HHH & \xrightarrow{\tau H} & HHH & \xrightarrow{H\tau} & HHH \\ H\tau \downarrow & & & & \downarrow \tau H \\ HHH & \xrightarrow{\tau H} & HHH & \xrightarrow{H\tau} & HHH \end{array}.$$

$\tau$  is called a *local prebraiding* provided it is a double entwining (see 6.1) and satisfies the Yang-Baxter equation.

**6.8. Doubling a bimonad.** Let  $\mathbf{H} = (H, m, e, \delta, \varepsilon)$  be a  $\tau$ -bimonad where  $\tau : HH \rightarrow HH$  is a local prebraiding. Then  $\mathbf{HH} = (HH, \bar{m}, \bar{e}, \bar{\delta}, \bar{\varepsilon})$  is a  $\bar{\tau}$ -bimonad with  $\bar{e} = ee$ ,  $\bar{\varepsilon} = \varepsilon\varepsilon$ ,

$$\begin{aligned} \bar{m} : HHHH &\xrightarrow{H\tau H} HHHH \xrightarrow{mm} HH, \\ \bar{\delta} : HH &\xrightarrow{\delta\delta} HHHH \xrightarrow{H\tau H} HHHH \end{aligned}$$

and double entwining

$$\bar{\tau} : HHHH \xrightarrow{H\tau H} HHHH \xrightarrow{\tau HH} HHHH \xrightarrow{HH\tau} HHHH \xrightarrow{H\tau H} HHHH.$$

**Proof.** We already know that  $(HH, \bar{m}, \bar{e})$  is a monad and that  $(HH, \bar{\delta}, \bar{\varepsilon})$  is a comonad. First we have to show that  $\bar{\tau}$  is a mixed distributive law from the monad  $(HH, \bar{m}, \bar{e})$  to the comonad  $(HH, \bar{\delta}, \bar{\varepsilon})$ , that is

$$\begin{aligned} HH\bar{e} &= \bar{\tau} \cdot \bar{e}HH, & HH\bar{\varepsilon} &= \bar{\varepsilon}HH \cdot \bar{\tau}, \\ HH\bar{m} \cdot \bar{\tau}HH &= \bar{\tau} \cdot \bar{m}HH, \\ HH\bar{\tau} \cdot \bar{\tau}HH &= \bar{\delta}HH \cdot \bar{\tau}. \end{aligned}$$

The first two equalities can be verified by placing the composites in suitable commutative diagrams. The second two identities are obtained by lengthy standard computations (as known for classical Hopf algebras).

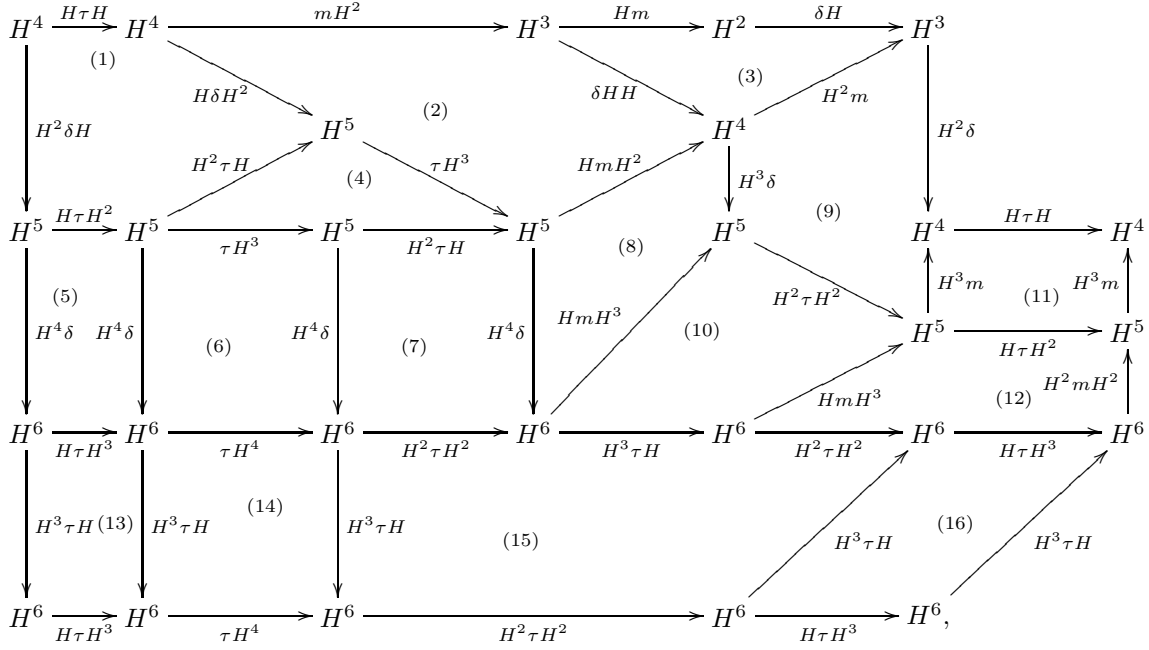
It remains to show that  $(HH, \bar{m}, \bar{e}, \bar{\delta}, \bar{\varepsilon})$  satisfies the conditions of Definition 4.1 with respect to  $\bar{\tau}$ . Again

$$\begin{aligned} \bar{\varepsilon} \cdot \bar{m} &= \varepsilon \cdot H\varepsilon \cdot HH\varepsilon\varepsilon = \bar{\varepsilon} \cdot HH\bar{\varepsilon}, \text{ and} \\ \bar{\delta} \cdot \bar{e} &= HHee \cdot He \cdot e = HH\bar{e}\bar{e} \end{aligned}$$

are shown by standard computations and

$$\bar{\varepsilon}\bar{e} = \varepsilon \cdot \varepsilon H \cdot eH \cdot e \stackrel{(4.2)}{=} \varepsilon \cdot e = 1.$$

To show that  $(HH, \bar{m}, \bar{e}, \bar{\delta}, \bar{\varepsilon}, \bar{\tau})$  satisfies (4.1), consider the diagram



in which *diagram* (1) commutes because  $\tau$  is a mixed distributive law and thus

$$H\tau \cdot \tau H \cdot H\delta = \delta H \cdot \tau;$$

the *diagrams* (2) and (9) commute by (4.1); the *diagrams* (3)-(8), (10), (11), (13), (14) and (16) commute by naturality; *diagram* (12) commutes because  $\tau$  is a mixed distributive law (hence  $Hm \cdot \tau H \cdot H\tau = \tau \cdot mH$ ); *diagram* (15) commutes by 6.7. By commutativity of the whole diagram,

$$\begin{aligned} \bar{\delta} \cdot \bar{m} &= H\tau H \cdot H^2\delta \cdot \delta H \cdot Hm \cdot mH^2 \cdot H\tau H \\ &= H^2m \cdot H^2mH^2 \cdot H^3\tau H \cdot H\tau H^3 \cdot H^2\tau H^2 \cdot \tau H^4 \cdot H\tau H^3 \cdot H^3\tau H \cdot H^4\delta \cdot H^2\delta H \\ &= HH\bar{\delta} \cdot \bar{\tau}HH \cdot HH\bar{m}, \end{aligned}$$

and hence  $\mathbf{HH} = (HH, \bar{m}, \bar{e}, \bar{\delta}, \bar{\varepsilon})$  is a  $\bar{\tau}$ -bimonad.  $\square$

**6.9. Opposite monad and comonad.** Let  $\tau : HH \rightarrow HH$  be a natural transformation satisfying the Yang-Baxter equation.

- (1) If  $(H, m, e)$  is a monad and  $\tau$  is monad distributive, then  $(H, m \cdot \tau, e)$  is also a monad and  $\tau$  is monad distributive for it.
- (2) If  $(H, \delta, \varepsilon)$  is a comonad and  $\tau$  is comonad distributive, then  $(H, \tau \cdot \delta, \varepsilon)$  is also a comonad and  $\tau$  is comonad distributive for it.



**Proof.** (1) To show that  $m \cdot \tau$  is associative construct the diagram

$$\begin{array}{ccccc}
 HHH & \xrightarrow{\tau_H} & HHH & \xrightarrow{m_H} & HH \\
 \downarrow H\tau & & \downarrow H\tau & & \downarrow \tau \\
 & (1) & HHH & & \\
 & & \downarrow \tau_H & & \\
 HHH & \xrightarrow{\tau_H} & HHH & \xrightarrow{H\tau} & HHH & \xrightarrow{Hm} & HH \\
 \downarrow Hm & & \downarrow mH & & \downarrow m \\
 HH & \xrightarrow{\tau} & HH & \xrightarrow{m} & H, \\
 & (3) & & (4) & 
 \end{array}$$

where the *rectangle* (1) is commutative by the YB-condition, (2) and (3) are commutative by the monad distributivity of  $\tau$ , and the *square* (4) is commutative by associativity of  $m$ . Now commutativity of the outer diagram shows associativity of  $m \cdot \tau$ .

From 2.5 we know that  $\tau \cdot e_H = He$  and  $\tau \cdot He = e_H$  and this implies that  $e$  is also the unit for  $(H, m \cdot \tau, e)$ .

The two pentagons for monad distributivity of  $\tau$  for  $(H, m \cdot m, e)$  can be read from the above diagram by combining the two top rectangles as well as the two left hand rectangles.

(2) The proof is dual to the proof of (1).  $\square$

**6.10. Opposite bimonad.** Let  $\tau : HH \rightarrow HH$  be a local prebraiding with  $\tau^2 = I$  and let  $\mathbf{H} = (H, m, e, \delta, \varepsilon)$  be a  $\tau$ -bimonad on  $\mathbb{A}$ . Then:

(1)  $\mathbf{H}' = (H, m \cdot \tau, e, \tau \cdot \delta, \varepsilon)$  is also a  $\tau$ -bimonad.

(2) If  $\mathbf{H}$  has an antipode  $S$  with  $\tau \cdot HS = SH \cdot \tau$  and  $\tau \cdot SH = HS \cdot \tau$ , then  $S$  is a  $\tau$ -bimonad morphism between the  $\tau$ -bimonads  $\mathbf{H}$  and  $\mathbf{H}'$ .

In this case  $S$  is an antipode for  $\mathbf{H}'$ .

**Proof.** (1) By (1), (2) in 6.9,  $\tau$  is a (co)monad distributive law from the (co)monad  $H$  to the (co)monad  $H'$ , and  $\varepsilon' \cdot e' = \varepsilon \cdot e = 1$  by (6.10). Moreover,

$$\varepsilon' \cdot m' = \varepsilon \cdot m \cdot \tau \stackrel{(6.10)}{=} \varepsilon \cdot H\varepsilon \cdot \tau \stackrel{2.4}{=} \varepsilon \cdot \varepsilon H = \varepsilon \cdot H\varepsilon = \varepsilon' \cdot H\varepsilon', \text{ and}$$

$$\delta' \cdot e' = \tau \cdot \delta \cdot e \stackrel{(6.10)}{=} \tau \cdot eH \cdot e \stackrel{2.1}{=} He \cdot e = eH \cdot e = e' \cdot H \cdot e'.$$

To prove compatibility for  $\mathbf{H}'$  we have to show the commutativity of the diagram

$$(6.17) \quad \begin{array}{ccccc}
 HH & \xrightarrow{m'} & H & \xrightarrow{\delta'} & HH \\
 \downarrow \delta'\delta' & & & & \uparrow m'm' \\
 HHHH & \xrightarrow{H\tau H} & HHHH & & 
 \end{array}$$

For this standard computations (from Hopf algebras) apply.

(2) By 6.6,  $S$  is a  $\tau$ -bimonad morphism from the  $\tau$ -bimonad  $\mathbf{H}$  to the  $\tau$ -bimonad  $\mathbf{H}'$ . To show that  $S$  is an antipode for  $\mathbf{H}'$  we need the equalities

$$m' \cdot SH \cdot \delta' = e' \cdot \varepsilon' = e \cdot \varepsilon \quad \text{and} \quad m' \cdot HS \cdot \delta' = e' \cdot \varepsilon' = e \cdot \varepsilon.$$

Since  $\tau \cdot SH = HS \cdot \tau$ , we have

$$m' \cdot SH \cdot \delta' = m \cdot \tau \cdot SH \cdot \tau \cdot \delta = m \cdot HS \cdot \tau \cdot \tau \cdot \delta \stackrel{\tau^2=1}{=} m \cdot HS \cdot \delta = e \cdot \varepsilon.$$

Since  $\tau \cdot HS = SH \cdot \tau$ , we have

$$m' \cdot HS \cdot \delta' = m \cdot \tau \cdot HS \cdot \tau \cdot \delta = m \cdot SH \cdot \tau \cdot \tau \cdot \delta \stackrel{\tau^2=1}{=} m \cdot SH \cdot \delta = e \cdot \varepsilon.$$

□

As we have seen in Theorem 5.6, the existence of an antipode for a bimonad  $\mathbf{H}$  on a category  $\mathbb{A}$  is equivalent to the comparison functor being an equivalence provided  $\mathbb{A}$  is Cauchy complete and  $H$  preserves colimits. Given a local prebraiding the latter condition on  $H$  can be replaced by conditions on the antipode (compare [3, Theorem 3.4], [4, Lemma 4.2] for the situation in braided monoidal category).

**6.11. Antipode and equivalence.** *Let  $\tau : HH \rightarrow HH$  be a local prebraiding and let  $\mathbf{H} = (H, m, e, \delta, \varepsilon)$  be a  $\tau$ -bimonad on a category  $\mathbb{A}$  in which idempotents split. Consider the category of bimodules*

$$\mathbb{A}_H^H = \mathbb{A}_{\underline{H}}^{\overline{H}}(\tilde{\tau}),$$

where  $\tilde{\tau} = mH \cdot H\tau \cdot \delta H$  (see 6.3, 6.5).

*If  $\mathbf{H}$  has an antipode  $S$  such that  $\tau \cdot SH = HS \cdot \tau$  and  $\tau \cdot HS = SH \cdot \tau$ , then the comparison functor  $K_{\underline{H}} : \mathbb{A} \rightarrow \mathbb{A}_{\underline{H}}^H$  is an equivalence of categories.*

**Proof.** We know that the functor  $K_{\underline{H}}$  has a right adjoint if for each  $(a, h_a, \theta_a) \in \mathbb{A}_{\underline{H}}^H$ , the equaliser of the  $(a, h_a, \theta_a)$ -component of the pair of functors

$$(6.18) \quad U_{\underline{H}} U^{\widehat{H}} \xrightleftharpoons[\beta_{U_{\underline{H}}} U^{\widehat{H}}]{U_{\underline{H}} U^{\widehat{H}} e_{\widehat{H}}} U_{\underline{H}} \widehat{H} U^{\widehat{H}} = U_{\underline{H}} U^{\widehat{H}} \phi^{\widehat{H}} U^{\widehat{H}}$$

exists. Here  $e_{\widehat{H}} : 1 \rightarrow \phi^{\widehat{H}} U^{\widehat{H}}$  is the unit of the adjunction  $U^{\widehat{H}} \dashv \phi^{\widehat{H}}$  and  $\beta_{U_{\underline{H}}}$  is the composite

$$U_{\underline{H}} \xrightarrow{e_{\underline{H}} U_{\underline{H}}} U_{\underline{H}} \phi_{\underline{H}} U_{\underline{H}} \xrightarrow{U_{\underline{H}}(t_{K_{\underline{H}}})} U_{\underline{H}} \widehat{H}.$$

Using the fact that for any  $(a, h_a) \in \mathbb{A}_{\underline{H}}$ ,

$$(t_{K_{\underline{H}}})_{(a, h_a)} = H(h_a) \cdot \delta_a \quad \text{and}$$

$$H(h_a) \cdot \delta_a \cdot e_a = H(h_a) \cdot H(e_a) \cdot e_a = e_a,$$

it is not hard to show that the  $(a, h_a, \theta_a)$ -component of Diagram 6.18 is the pair

$$a \xrightleftharpoons[\theta_a]{e_a} H(a).$$

Thus,  $K_{\underline{H}}$  has a right adjoint if for each  $(a, h_a, \theta_a) \in \mathbb{A}_{\underline{H}}^H$ , the equaliser of the pair of morphisms  $(e_a, \theta_a)$  exists.

Suppose now that  $\mathbf{H}$  has an antipode  $S : H \rightarrow H$ . For each  $(a, h_a, \theta_a) \in \mathbb{A}_{\underline{H}}^H$ , consider the composite  $q_a = h_a \cdot S_a \cdot \theta_a : a \rightarrow a$ . By a (tedious) standard computation - applying 6.15, 6.6, 2.4 - one can show

$$e_a \cdot q_a = \theta_a \cdot q_a \text{ and } q_a \cdot q_a = q_a.$$

**6.12. Remark.** Dually, one can prove that for each  $(a, h_a, \theta_a) \in \mathbb{A}_{\underline{H}}^H$ ,  $q_a \cdot \varepsilon_a = q_a \cdot h_a$ , thus  $i_a \cdot \bar{q}_a \cdot \varepsilon_a = i_a \cdot \bar{q}_a \cdot h_a$ , and since  $i_a$  is a (split) monomorphism, it follows that  $\bar{q}_a \cdot \varepsilon_a = \bar{q}_a \cdot h_a$ .

Since idempotents split in  $\mathbb{A}$ , there exist morphisms  $i_a : \bar{a} \rightarrow a$  and  $\bar{q}_a : a \rightarrow \bar{a}$  such that  $\bar{q}_a \cdot i_a = 1_a$  and  $i_a \cdot \bar{q}_a = q_a$ . Since  $\bar{q}_a$  is a (split) epimorphism and since  $e_a \cdot i_a \cdot \bar{q}_a = e_a \cdot q_a = \theta_a \cdot q_a = \theta \cdot i_a \cdot \bar{q}_a$ , it follows that

$$(6.19) \quad e_a \cdot i_a = \theta_a \cdot i_a.$$

Using this equality it is straightforward to show that the diagram

$$(6.20) \quad \begin{array}{ccccc} & \xleftarrow{\bar{q}_a} & & \xleftarrow{h_a \cdot S_a} & \\ \bar{a} & \xrightleftharpoons[i_a]{\bar{q}_a} & a & \xrightleftharpoons[\theta_a]{e_a} & H(a) \end{array}$$

is a split equaliser. Hence for any  $(a, h_a, \theta_a) \in \mathbb{A}_H^H$ , the equaliser of the pair of morphisms  $(e_a, \theta_a)$  exists, which implies that the functor  $K_{\underline{H}}$  has a right adjoint  $R_{\underline{H}} : \mathbb{A}_H^H \rightarrow \mathbb{A}$  which is given by

$$R_{\underline{H}}(a, H_a, \theta_a) = \bar{a}.$$

Since for any  $(a, h_a, \theta_a) \in \mathbb{A}_H^H$ ,

- $\delta_a \cdot e_a = e_{H(a)} \cdot e_a$  and  $\varepsilon_a \cdot e_a = 1$  by 6.2;
- $\varepsilon_{H(a)} \cdot \delta_a = 1$ , since  $(H, \varepsilon, \delta)$  is a comonad;
- $e_a \cdot \varepsilon_a = \varepsilon_{H(a)} \cdot e_{H(a)}$  by naturality,

we get a split equaliser diagram

$$\begin{array}{ccccc} & \xleftarrow{\varepsilon_a} & & \xleftarrow{H(\varepsilon_a)} & \\ a & \xrightleftharpoons[e_a]{\varepsilon_a} & H(a) & \xrightleftharpoons[\delta_a]{e_{H(a)}} & H^2(a) \end{array}$$

This is preserved by any functor, and since  $R_{\overline{H}}(H(a), m_a, \delta_a)$  is the equaliser of the pair of morphisms  $(e_{H(a)}, \delta_a)$ , in particular  $a \simeq R_{\overline{H}}(H(a), m_a, \delta_a) = R_{\overline{H}}(K_{\overline{H}}(a))$ . Thus  $R_{\overline{H}}K_{\overline{H}} \simeq 1$ .

For any  $(a, h_a, \theta_a) \in \mathbb{A}_H^H$ , write  $\alpha_a$  for the composite  $h_a \cdot H(i_a) : H(\bar{a}) \rightarrow a$ . We claim that  $\alpha_a$  is a morphism in  $\mathbb{A}_H^H$  from  $K_{\underline{H}}(\bar{a}) = (H(\bar{a}), m_{\bar{a}}, \delta_{\bar{a}})$  to  $(a, h_a, \theta_a)$ . Indeed, we have

$$\begin{aligned} \alpha_a \cdot m_{\bar{a}} &= h_a \cdot H(i_a) \cdot m_{\bar{a}} \\ \text{naturality} &= h_a \cdot m_a \cdot H^2(i_a) \\ (a, h_a) \in \mathbb{A}_{\underline{H}} &= h_a \cdot H(h_a) \cdot H^2(i_a) = h_a \cdot H(H(h_a) \cdot i_a) = h_a \cdot H(\alpha_a), \end{aligned}$$

and this just means that  $\alpha_a$  is a morphism in  $\mathbb{A}_{\underline{H}}$  from  $(H(\bar{a}), m_{\bar{a}})$  to  $(a, h_a)$ .

Next - using 6.15, 6.19 - we compute

$$\theta_a \cdot \alpha_a = H(\alpha_a) \cdot \delta_{\bar{a}}.$$

Thus,  $\alpha_a$  is a morphism in  $\mathbb{A}^{\overline{H}}$  from  $(H(\bar{a}), \delta_{\bar{a}})$  to  $(a, \delta_a)$ , and hence  $\alpha_a$  is a morphism in  $\mathbb{A}_H^H$  from  $K_{\underline{H}}(\bar{a}) = (\bar{a}, m_{\bar{a}}, \delta_{\bar{a}})$  to  $(a, h_a, \theta_a)$ .

Similarly it is proved that the composite  $\beta_a = H(\bar{q}_a) \cdot \theta_a : a \rightarrow H(\bar{a})$  is a morphism in  $\mathbb{A}^{\overline{H}}$  from  $(a, h_a, \delta_a)$  to  $(H(\bar{a}), m_{\bar{a}}, \delta_{\bar{a}})$  and a further calculation yields

$$\alpha_a \cdot \beta_a = 1_a \text{ and } \beta_a \cdot \alpha_a = 1_{H(\bar{a})}.$$

Hence we have proved that for any  $(a, h_a, \theta_a) \in \mathbb{A}_H^H$ ,  $\alpha_a$  is an isomorphism in  $\mathbb{A}_H^H$ , and using the fact that the same argument as in Remark 2.4 in [12] shows that  $\alpha_a$  is the counit of the adjunction  $K_{\underline{H}} \dashv R_{\underline{H}}$ , one concludes that  $K_{\underline{H}}R_{\underline{H}} \simeq 1$ . Thus the functor  $K_{\underline{H}}$  is an equivalence of categories. This completes the proof.  $\square$

For an example, let  $\mathcal{V} = (\mathbb{V}, \otimes, I, \sigma)$  be a braided monoidal category and  $\mathbf{H} = (H, m, e, \delta, \varepsilon)$  a bialgebra in  $\mathcal{V}$ . Then

$$(H \otimes -, m \otimes -, e \otimes -, \delta \otimes -, \varepsilon \otimes -, \tau = \sigma_{H, H} \otimes -)$$

is a bimonad on  $\mathbb{V}$ , and it is easy to see that the category  $\mathbb{V}_H^H$  of Hopf modules is just the category  $\mathbb{V}_{H \otimes -}^{\overline{H \otimes -}}(\bar{\tau}) = \mathbb{V}_{H \otimes -}^{H \otimes -}$ .

**6.13. Theorem.** *Let  $\mathcal{V} = (\mathbb{V}, \otimes, I, \sigma)$  be a braided monoidal category such that idempotents split in  $\mathbb{V}$ . Then for any bialgebra  $\mathbf{H} = (H, m, e, \delta, \varepsilon)$  in  $\mathcal{V}$ , the following are equivalent:*

- (a)  $\mathbf{H}$  has an antipode;
- (b) the comparison functor

$$K_H : \mathbb{V} \rightarrow \mathbb{V}_H^H, \quad V \mapsto (H \otimes V, m \otimes V, \delta \otimes V), \quad f \mapsto H \otimes f,$$

is an equivalence of categories.

## 7. ADJOINTS OF BIMONADS

This section deals with the transfer of properties of monads and comonads to adjoint (endo-)functors. The relevance of this interplay was already observed by Eilenberg and Moore in [11]. An effective formalism to handle this was developed for adjunctions in 2-categories and is nicely presented in Kelly and Street [16]. For our purpose we only need this for the 2-category of categories and for convenience we recall the basic facts of this situation here.

**7.1. Adjunctions.** Let  $L : \mathbb{A} \rightarrow \mathbb{B}$ ,  $R : \mathbb{B} \rightarrow \mathbb{A}$  be an adjoint pair of functors with unit and counit  $\eta, \varepsilon$ , and  $L' : \mathbb{A}' \rightarrow \mathbb{B}'$ ,  $R' : \mathbb{B}' \rightarrow \mathbb{A}'$  be an adjoint pair of functors with unit and counit  $\eta', \varepsilon'$ . Given any functors  $F : \mathbb{A} \rightarrow \mathbb{A}'$  and  $G : \mathbb{B} \rightarrow \mathbb{B}'$ , there is a bijection between natural transformations

$$\alpha : L'F \rightarrow GL \quad \text{and} \quad \bar{\alpha} : FR \rightarrow R'G$$

where  $\bar{\alpha}$  is obtained as the composite

$$FR \xrightarrow{\eta'FR} R'L'FR \xrightarrow{R'\alpha R} R'GLR \xrightarrow{R'G\varepsilon} R'G,$$

and  $\alpha$  is given as the composite

$$L'F \xrightarrow{L'F\eta} L'FRL \xrightarrow{L'\bar{\alpha}L} L'R'GL \xrightarrow{\varepsilon'GF} GL.$$

In this situation,  $\alpha$  and  $\bar{\alpha}$  are called *mates* under the given adjunction and this is denoted by  $\alpha \dashv \bar{\alpha}$ . It is nicely displayed in the diagram

$$\begin{array}{ccccc} \mathbb{A} & \xrightarrow{L} & \mathbb{B} & \xrightarrow{R} & \mathbb{A} \\ F \downarrow & \nearrow \alpha & \downarrow G & \nwarrow \bar{\alpha} & \downarrow F \\ \mathbb{A}' & \xrightarrow{L'} & \mathbb{B}' & \xrightarrow{R'} & \mathbb{A}' \end{array}$$

Given further

(i) adjunctions  $\tilde{L} : \mathbb{C} \rightarrow \mathbb{A}$ ,  $\tilde{R} : \mathbb{A} \rightarrow \mathbb{C}$  and  $\tilde{L}' : \mathbb{C}' \rightarrow \mathbb{A}'$ ,  $\tilde{R}' : \mathbb{A}' \rightarrow \mathbb{C}'$  and a functor  $H : \mathbb{C} \rightarrow \mathbb{C}'$ , or

(ii) an adjunction  $L'' : \mathbb{A}'' \rightarrow \mathbb{B}''$ ,  $R'' : \mathbb{B}'' \rightarrow \mathbb{A}''$  and functors  $F' : \mathbb{A}' \rightarrow \mathbb{A}''$  and  $G' : \mathbb{B}' \rightarrow \mathbb{B}''$ , we get the diagram

$$\begin{array}{ccccccc} \mathbb{C} & \xrightarrow{\tilde{L}} & \mathbb{A} & \xrightarrow{L} & \mathbb{B} & \xrightarrow{R} & \mathbb{A} & \xrightarrow{\tilde{R}} & \mathbb{C} \\ H \downarrow & \nearrow \gamma & \downarrow F & \nearrow \alpha & \downarrow G & \nwarrow \bar{\alpha} & \downarrow F & \nwarrow \bar{\gamma} & \downarrow H \\ \mathbb{C}' & \xrightarrow{\tilde{R}'} & \mathbb{A}' & \xrightarrow{L'} & \mathbb{B}' & \xrightarrow{R'} & \mathbb{A}' & \xrightarrow{\tilde{L}'} & \mathbb{C}' \\ & & \downarrow F' & \nearrow \beta & \downarrow G' & \nwarrow \bar{\beta} & \downarrow F' & & \\ & & \mathbb{A}'' & \xrightarrow{L''} & \mathbb{B}'' & \xrightarrow{R''} & \mathbb{A}'' & & \end{array}$$

yielding the mates

$$(M1) \quad L''F'F \xrightarrow{\beta F} G'L'F \xrightarrow{G'\alpha} G'GL \dashv F'FG \xrightarrow{F'\bar{\alpha}} F'R'G \xrightarrow{\bar{\beta}G} R''G'G,$$

$$(M2) \quad L'\tilde{L}'H \xrightarrow{L'\beta} L'F\tilde{L} \xrightarrow{\alpha\tilde{L}} LG\tilde{L} \dashv H\tilde{R}R \xrightarrow{\bar{\beta}G} \tilde{R}'R'G \xrightarrow{\tilde{R}'\bar{\beta}} \tilde{R}'R'G.$$

**7.2. Properties of mates.** Let  $L, L' : \mathbb{A} \rightarrow \mathbb{B}$  be functors with right adjoints  $R, R'$ , respectively, and  $\alpha : L' \rightarrow L$  a natural transformation.

(i) If  $L'' : \mathbb{A} \rightarrow \mathbb{B}$  is a functor with right adjoint  $R''$  and  $\beta : L'' \rightarrow L'$  a natural transformation, then

$$\alpha \cdot \beta \dashv \bar{\beta} \cdot \bar{\alpha}.$$

(ii) If  $\tilde{L} : \mathbb{C} \rightarrow \mathbb{A}$  is a functor with right adjoint  $\tilde{R}$ , then

$$(\alpha_{L'} : L'\tilde{L} \rightarrow L\tilde{L}) \dashv (\tilde{R}\bar{\alpha} : \tilde{R}R \rightarrow \tilde{R}R').$$

(iii) If  $L^o : \mathbb{B} \rightarrow \mathbb{C}$  is a functor with right adjoint  $R^o$ , then

$$(L^o\alpha : L^oL' \rightarrow L^oL) \dashv (\bar{\alpha}R^o : RR^o \rightarrow R'R^o).$$

**Proof.** (i) is a special case of 7.1(M1).

(ii) follows from 7.1(M2) by putting  $\mathbb{A}' = \mathbb{A}$ ,  $\mathbb{B}' = \mathbb{B}$ ,  $\mathbb{C}' = \mathbb{C}$  and  $H' = H$ .

(iii) is derived by applying 7.1 to the diagram

$$\begin{array}{ccccccc} \mathbb{A} & \xrightarrow{L} & \mathbb{B} & \xrightarrow{L^o} & \mathbb{C} & \xrightarrow{R^o} & \mathbb{B} & \xrightarrow{R} & \mathbb{A} \\ \parallel & \nearrow \alpha & \parallel & \nearrow I & \parallel & \nearrow I & \parallel & \nearrow \bar{\alpha} & \parallel \\ \mathbb{A} & \xrightarrow{\tilde{L}'} & \mathbb{B} & \xrightarrow{L^o} & \mathbb{C} & \xrightarrow{R^o} & \mathbb{B} & \xrightarrow{R'} & \mathbb{A}. \end{array}$$

□

As observed by Eilenberg and Moore in [11, Section 3], for a left adjoint endofunctor which is a monad, the right adjoint (if it exists) is a comonad (and vice versa). The techniques outlined above provide a convenient and effective way to describe this transition and to prove related properties. Recall that for any endofunctor  $L : \mathbb{A} \rightarrow \mathbb{A}$  with right adjoint  $R$ , for a positive integer  $n$ , the powers  $L^n$  have the right adjoints  $R^n$ .

**7.3. Adjoints of monads and comonads.** Let  $L : \mathbb{A} \rightarrow \mathbb{A}$  be an endofunctor with right adjoint  $R$ .

(1) If  $\underline{L} = (L, m_L, e_L)$  is a monad, then  $\bar{R} = (R, \delta_R, \varepsilon_R)$  is a comonad, where  $\delta_R, \varepsilon_R$  are the mates of  $m_L, e_L$  in the diagrams

$$\begin{array}{ccc} \mathbb{A} & \xrightarrow{L} & \mathbb{A} & \xrightarrow{R} & \mathbb{A} \\ \parallel & \nearrow e_L & \parallel & \nearrow \varepsilon_R & \parallel \\ \mathbb{A} & \xrightarrow{I} & \mathbb{A} & \xrightarrow{I} & \mathbb{A}, \end{array} \quad \begin{array}{ccc} \mathbb{A} & \xrightarrow{L} & \mathbb{A} & \xrightarrow{R} & \mathbb{A} \\ \parallel & \nearrow m_L & \parallel & \nearrow \delta_R & \parallel \\ \mathbb{A} & \xrightarrow{HH} & \mathbb{A} & \xrightarrow{RR} & \mathbb{A}. \end{array}$$

(2) If  $\bar{L} = (L, \delta_L, \varepsilon_L)$  is a comonad, then  $\underline{R} = (R, m_R, e_R)$  is a monad where  $m_R, e_R$  are the mates of  $\delta_L, \varepsilon_L$  in the diagrams

$$\begin{array}{ccc} \mathbb{A} & \xrightarrow{I} & \mathbb{A} & \xrightarrow{I} & \mathbb{A} \\ \parallel & \nearrow \varepsilon_L & \parallel & \nearrow e_R & \parallel \\ \mathbb{A} & \xrightarrow{L} & \mathbb{A} & \xrightarrow{R} & \mathbb{A}, \end{array} \quad \begin{array}{ccc} \mathbb{A} & \xrightarrow{LL} & \mathbb{A} & \xrightarrow{RR} & \mathbb{A} \\ \parallel & \nearrow \delta_L & \parallel & \nearrow m_R & \parallel \\ \mathbb{A} & \xrightarrow{L} & \mathbb{A} & \xrightarrow{R} & \mathbb{A}. \end{array}$$

**Proof.** (1) Since  $e_L \dashv \varepsilon_R$  and  $m_L \dashv \delta_R$ , it follows from 7.2 (ii) and (iii) that

$$Le_L \dashv \varepsilon_R R, \quad e_L L \dashv R \varepsilon_R, \quad m_L L \dashv R \delta_R, \quad L m_L \dashv \delta_R R.$$

Applying 7.2 (i) now yields

$$\begin{aligned} m_L \cdot Le_L \dashv \varepsilon_R R \cdot \delta_R, & \quad m_L \cdot e_L L \dashv R \varepsilon_R \cdot \delta_R, \\ m_L \cdot m_L L \dashv R \delta_R \cdot \delta_R, & \quad m_L \cdot L m_L \dashv \delta_R R \cdot \delta_R. \end{aligned}$$

Since  $\underline{L}$  is a monad we have  $m_L \cdot e_L L = m_L \cdot Le_L = I$  and  $m_L \cdot m_L L = m_L \cdot L m_L$ , implying

$$\varepsilon_R R \cdot \delta_R = R \varepsilon_R \cdot \delta_R = I \quad \text{and} \quad R \delta_R \cdot \delta_R = \delta_R R \cdot \delta_R.$$

This shows that  $\overline{R} = (R, \delta_R, \varepsilon_R)$  is a comonad.

The proof of (2) is similar. □

The methods under consideration also apply to the natural transformations  $LL \rightarrow LL$  which were basic for the definition and investigation of bimonads in previous sections. The following results were obtained in cooperation with Gabriella Böhm and Tomasz Brzeziński.

**7.4. Adjointness and distributive laws.** *Let  $L : \mathbb{A} \rightarrow \mathbb{A}$  be an endofunctor with right adjoint  $R$  and a natural transformation  $\lambda_L : LL \rightarrow LL$ . This yields a mate  $\lambda_R : RR \rightarrow RR$  in the diagram*

$$\begin{array}{ccccc} \mathbb{A} & \xrightarrow{LL} & \mathbb{A} & \xrightarrow{RR} & \mathbb{A} \\ \parallel & \nearrow \lambda_L & \parallel & \nwarrow \lambda_R & \parallel \\ \mathbb{A} & \xrightarrow{LL} & \mathbb{A} & \xrightarrow{RR} & \mathbb{A} \end{array}$$

with the following properties:

- (1)  $L\lambda_L \dashv \lambda_R R$  and  $\lambda_L L \dashv R\lambda_R$ .
- (2)  $\lambda_L$  satisfies the Yang-Baxter equation if and only if  $\lambda_R$  does.
- (3)  $\lambda_L^2 = I$  if and only if  $\lambda_R^2 = I$ .
- (4) If  $\underline{L} = (L, m_L, e_L)$  is a monad and  $\lambda_L$  is monad distributive, then  $\lambda_R$  is comonad distributive for the comonad  $\overline{R} = (R, \delta_R, \varepsilon_R)$ .
- (5) If  $\overline{L} = (L, \delta_L, \varepsilon_L)$  is a comonad and  $\lambda_L$  is comonad distributive, then  $\lambda_R$  is monad distributive for the monad  $\underline{R} = (R, m_R, e_R)$ .

**Proof.** (1) follows from 7.2, (ii) and (iii). The remaining assertions follow by (1) and the identities in the proof of 7.3. □

Recall from Definition 4.1 that a *bimonad*  $H$  is a monad and a comonad with compatibility conditions involving an entwining  $\lambda_H : HH \rightarrow HH$ .

**7.5. Adjoints of bimonads.** *Let  $\mathbf{H}$  be a monad  $\underline{H} = (H, m_H, e_H)$  and a comonad  $\overline{H} = (H, \delta_H, \varepsilon_H)$  on the category  $\mathbb{A}$ . Then a right adjoint  $R$  of  $H$  induces a monad  $\underline{R} = (R, m_R, e_R)$  and a comonad  $\overline{R} = (R, \delta_R, \varepsilon_R)$  (see 7.3) and*

- (1)  $\mathbf{H} = (\underline{H}, \overline{H})$  is a bimonad with entwining  $\lambda_H : \underline{H}\overline{H} \rightarrow \overline{H}\underline{H}$  if and only if  $\mathbf{R} = (\underline{R}, \overline{R})$  is a bimonad with entwining  $\lambda_R : \underline{R}\overline{R} \rightarrow \overline{R}\underline{R}$ .
- (2)  $\mathbf{H} = (\underline{H}, \overline{H})$  is a bimonad with entwining  $\lambda'_H : \overline{H}\underline{H} \rightarrow \underline{H}\overline{H}$  if and only if  $\mathbf{R} = (\underline{R}, \overline{R})$  is a bimonad with entwining  $\lambda'_R : \overline{R}\underline{R} \rightarrow \underline{R}\overline{R}$ .
- (3) If  $\mathbf{H} = (\underline{H}, \overline{H}, \lambda_H)$  is a bimonad with antipode, then  $\mathbf{R} = (\underline{R}, \overline{R}, \lambda_R)$  is a bimonad with antipode (Hopf monad).

**Proof.** (1) With arguments similar to those in the proof of 7.4 we get that  $\lambda_R$  is an entwining from  $\overline{R}$  to  $\underline{R}$ . It remains to show the properties required in Definition 4.1. From 7.2(i) we know that

$$\begin{aligned} \varepsilon_H \cdot H\varepsilon_H \dashv e_R R \cdot e_R, \quad \varepsilon_H \cdot m_H \dashv \delta_R \cdot e_R, \\ \delta_H \cdot e_H \dashv \varepsilon_R \cdot m_R, \quad e_H H \cdot e_H \dashv \varepsilon_R \cdot R\varepsilon_R, \quad \varepsilon_H \cdot e_H \dashv \varepsilon_R \cdot e_R. \end{aligned}$$

Thus the equalities

$$\varepsilon_H \cdot H\varepsilon_H = \varepsilon_H \cdot m_H, \quad \delta_H \cdot e_H = \varepsilon_H H \cdot e_H, \quad \varepsilon_H \cdot e_H = I$$

hold if and only if

$$e_R R \cdot e_R = \delta_R \cdot e_R, \quad \varepsilon_R \cdot m_R = \varepsilon_R \cdot R\varepsilon_R, \quad \varepsilon_R \cdot e_R = I.$$

The transfer of the compatibility between product and coproduct 4.1 is seen from the corresponding diagrams

$$\begin{array}{ccc} HH & \xrightarrow{m_H} & H \xrightarrow{\delta_H} HH \\ H\delta_H \downarrow & & \uparrow Hm_H \\ HHH & \xrightarrow{\lambda_H H} & HHH, \end{array} \quad \begin{array}{ccc} RR & \xleftarrow{\delta_R} & R \xleftarrow{m_R} RR \\ m_{RR} \uparrow & & \downarrow \delta_{RR} \\ RRR & \xleftarrow{R\lambda_R} & RRR. \end{array}$$

The proof of (2) is similar.

(3) By 5.5, the existence of an antipode is equivalent to the bijectivity of the morphism

$$\gamma_H = Hm_H \cdot \delta_H H : HH \rightarrow HH.$$

Since  $\delta_H H \dashv Rm_R$  and  $Hm_H \dashv \delta_R R$ ,  $\gamma_H$  is an isomorphism if and only if  $\gamma_R = Rm_R \cdot \delta_R R$  is an isomorphism.  $\square$

Functors with right (resp. left) adjoints preserve colimits (resp. limits) and thus 5.6 and 7.5 imply:

**7.6. Hopf monads with adjoints.** Assume the category  $\mathbb{A}$  to admit limits or colimits. Let  $\mathbf{H} = (H, m_H, e_H, \delta_H, \varepsilon_H, \lambda_H)$  be a bimonad on  $\mathbb{A}$  with a right adjoint bimonad  $\mathbf{R} = (R, m_R, e_R, \delta_R, \varepsilon_R, \lambda_R)$ . Then the following are equivalent:

- (a) the comparison functor  $K_H : \mathbb{A} \rightarrow \mathbb{A}_{\overline{H}}^{\underline{H}}(\lambda_H)$  is an equivalence;
- (b) the comparison functor  $K_R : \mathbb{A} \rightarrow \mathbb{A}_{\underline{R}}^{\overline{R}}(\lambda_R)$  is an equivalence;
- (c)  $\mathbf{H}$  has an antipode;
- (d)  $\mathbf{R}$  has an antipode.

Finally we observe that local prebraidings are also transferred to the adjoint functor.

**7.7. Adjointness of  $\tau$ -bimonads.** Let  $H$  be a monad  $\underline{H} = (H, m_H, e_H)$  and a comonad  $\overline{H} = (H, \delta_H, \varepsilon_H)$  on the category  $\mathbb{A}$  with a right adjoint  $R$ .

If  $\mathbf{H} = (\underline{H}, \overline{H})$  is a bimonad with double entwining  $\tau_H : HH \rightarrow HH$ , then  $\mathbf{R} = (\underline{R}, \overline{R})$  is a bimonad with double entwining  $\tau_R : RR \rightarrow RR$ .

Moreover,  $\tau_H$  satisfies the Yang-Baxter equation if and only if so does  $\tau_R$ .

**Proof.** Most of the assertions follow immediately from 7.4 and 7.5.

It remains to verify the compatibility condition 6.9. For this observe that from 7.2(i) we get

$$\delta_H \delta_H \dashv m_H m_H, \quad H\tau_H H \dashv R\tau_R R, \quad m_H m_H \dashv \delta_R \delta_R,$$

and hence

$$m_H m_H \cdot \tau_H H \cdot \delta_H \delta_H \dashv m_R m_R \cdot R\tau_R R \cdot \delta_R \delta_R \text{ and } \delta_H \cdot m_H \dashv \delta_R \cdot m_R.$$

It follows that  $\mathbf{H}$  satisfies 6.9 if and only if so does  $\mathbf{R}$ .  $\square$

**7.8. Dual Hopf algebras.** Let  $B$  be a module over a commutative ring  $R$ .  $B$  is a Hopf algebra if and only if the endofunctor  $B \otimes_R -$  on the category of  $R$ -modules is a Hopf monad. By 7.5,  $B \otimes_R -$  is a bimonad (with antipode) if and only if its right adjoint functor  $\text{Hom}_R(B, -)$  is a bimonad (with antipode). This situation is considered in more detail in [5].

If  $B$  is finitely generated and projective as an  $R$ -module and  $B^* = \text{Hom}_R(B, R)$ , then  $\text{Hom}_R(B, -) \simeq B^* \otimes_R -$  and we obtain the familiar result that  $B$  is a Hopf algebra if and only if  $B^*$  is.

**7.9. Characterisations of groups.** For any set  $G$ , the endofunctor  $G \times - : \mathbf{Set} \rightarrow \mathbf{Set}$  is a Hopf bimonad on the category of sets if and only if  $G$  has a group structure (e.g. [33, 5.20]). Since the functor  $\text{Map}(G, -)$  is right adjoint to  $G \times -$ , it follows from 7.6 that a set  $G$  is a group if and only if the functor  $\text{Map}(G, -) : \mathbf{Set} \rightarrow \mathbf{Set}$  is a Hopf monad.

**Acknowledgements.** The authors want to express their thanks to Gabriella Böhm and Tomasz Brzeziński for inspiring discussions and helpful comments. The research was started during a visit of the first author at the Department of Mathematics at the Heinrich Heine University of Düsseldorf supported by the German Research Foundation (DFG). He is grateful to his hosts for the warm hospitality and to the DFG for the financial help.

## REFERENCES

- [1] Barr, M., *Composite cotriples and derived functors*, in: Sem. Triples Categor. Homology Theory, Springer LN Math. 80, 336-356 (1969)
- [2] Beck, J., *Distributive laws*, in: Seminar on Triples and Categorical Homology Theory, B. Eckmann (ed.), Springer LNM 80, 119-140 (1969)
- [3] Bespalov, Y. and Drabant, B., *Hopf (bi-)modules and crossed modules in braided monoidal categories*, J. Pure Appl. Algebra 123(1-3), 105-129 (1998)
- [4] Bespalov, Y., Kerler, Th., Lyubashenko V. and Turaev, V., *Integrals for braided Hopf algebras*, J. Pure Appl. Algebra 148(2), 113-164 (2000)
- [5] Böhm, G., Brzeziński, T. and Wisbauer, R., *Monads and comonads in module categories*, preprint
- [6] Borceux, F. and Dejean, D., *Cauchy completion in category theory*, Cah. Topol. Géom. Différ. Catégoriques 27, 133-146 (1986)
- [7] Bruguières, A. and Virelizier, A., *Hopf monads*, Adv. Math. 215(2), 679-733 (2007)
- [8] Brzeziński, T. and Wisbauer, R., *Corings and Comodules*, London Math. Soc. Lecture Note Series 309, Cambridge University Press (2003)
- [9] Dubuc, E., *Adjoint triangles*, Rep. Midwest Category Semin. 2, Lect. Notes Math. 61, 69-91 (1968)
- [10] Dubuc, E., *Kan extensions in enriched category theory*, Lecture Notes in Mathematics 145, Berlin-Heidelberg-New York: Springer-Verlag (1970)
- [11] Eilenberg, S. and Moore, J.C., *Adjoint functors and triples*, Ill. J. Math. 9, 381-398 (1965)
- [12] Gómez-Torrecillas, J., *Comonads and Galois corings*, Appl. Categ. Struct. 14(5-6), 579-598 (2006)
- [13] Gumm, H.P., *Universelle Coalgebra*, in: Allgemeine Algebra, Ihringer, Th., Berliner Stud. zur Math., Band 10, 155-207, Heldermann Verlag (2003)
- [14] Hardie, K.A., *Projectivity and injectivity relative to a functor*, Math. Colloq., Univ. Cape Town 10, 68-80 (1957/76)
- [15] Janelidze, G. and W. Tholen, W., *Facets of Descent, III : Monadic Descent for Rings and Algebras*, Appl. Categorical Structures 12, 461-476 (2004)
- [16] Kelly, G.M. and Street, R., *Review of the elements of 2-categories*, Category Sem., Proc., Sydney 1972/1973, Lect. Notes Math. 420, 75-103 (1974)
- [17] Loday, J.-L., *Generalized bialgebras and triples of operads*, arXiv:math/0611885
- [18] McCrudden, P., *Opmonoidal monads*, Theory Appl. Categ. 10, 469-485 (2002)
- [19] Mesablishvili, B., *Descent in categories of (co)algebras*, Homology, Homotopy and Applications 7, 1-8 (2005)
- [20] Mesablishvili, B., *Monads of effective descent type and comonadicity*, Theory Appl. Categ. 16, 1-45 (2006)



- [21] Mesablishvili, B., *Entwining Structures in Monoidal Categories*, J. Algebra 319(6), 2496-2517 (2008)
- [22] Moerdijk, I., Monads on tensor categories, J. Pure Appl. Algebra 168(2-3), 189-208 (2002)
- [23] Power, J. and Watanabe, H., *Combining a monad and a comonad*, Theor. Comput. Sci. 280(1-2), 137-162 (2002)
- [24] Schubert, H., *Categories*, Berlin-Heidelberg-New York, Springer-Verlag (1972)
- [25] Škoda, Z., *Distributive laws for actions of monoidal categories*, arXiv:math.CT/0406310 (2004)
- [26] Sobral, M., *Restricting the comparison functor of an adjunction to projective objects*, Quaest. Math. 6, 303-312 (1983)
- [27] Street, R., *Frobenius monads and pseudomonoids*, J. Math. Phys. 45(10), 3930-3948 (2004)
- [28] Szlachányi, K., The monoidal Eilenberg-Moore construction and bialgebroids, J. Pure Appl. Algebra 182(2-3), 287-315 (2003)
- [29] Szlachányi, K., *Adjointable monoidal functors and quantum groupoids*, Caenepeel, S. (ed.) et al., Hopf algebras in noncommutative geometry and physics, Proc. conf. on Hopf algebras and quantum groups, Brussels 2002. Marcel Dekker. LN PAM 239, 291-307 (2005)
- [30] Takeuchi, M., *Survey of braided Hopf algebras*, in: New trends in Hopf algebra theory, Proc. Coll. Quantum Groups and Hopf Algebras, La Falda, Argentina 1999, Andruskiewitsch, N. et al.(ed.), Providence, RI: American Math. Soc., Contemp. Math. 267, 301-323 (2000)
- [31] Turi, D. and Plotkin, G., *Towards a mathematical operational semantics*, Proceedings 12th Ann. IEEE Symp. on Logic in Computer Science, LICS'97, Warsaw, Poland (1997)
- [32] Wisbauer, R., *On Galois comodules*, Commun. Algebra 34(7), 2683-2711 (2006)
- [33] Wisbauer, R., *Algebras versus coalgebras*, Appl. Categor. Struct. 16(1-2), 255-295 (2008)
- [34] Wolff, H., *V-Localizations and V-monads*. J. Algebra 24, 405-438 (1973)

#### Addresses:

Razmadze Mathematical Institute, Tbilisi 0193, Republic of Georgia  
 bachi@rmi.acnet.ge

Department of Mathematics of HHU, 40225 Düsseldorf, Germany  
 wisbauer@math.uni-duesseldorf.de